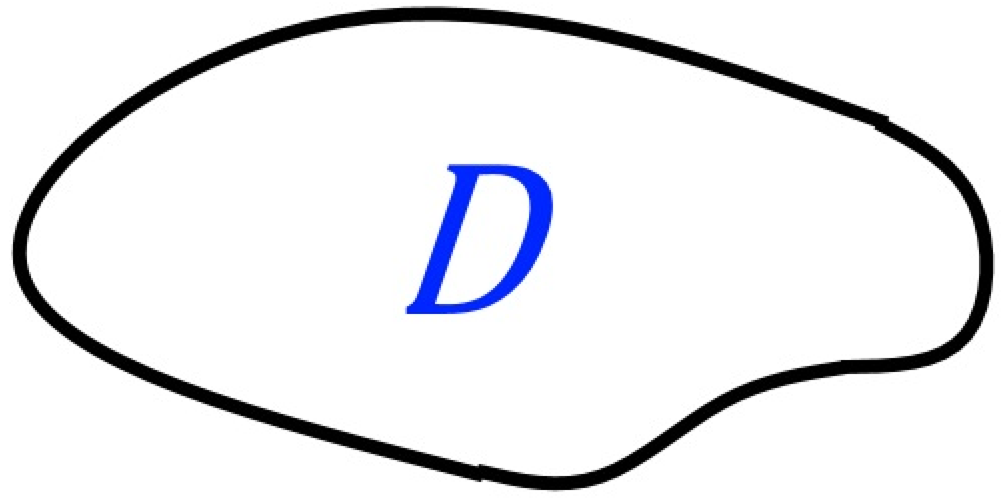
Schrödinger equation (a particle in a bounded domain):

$$i\,\psi_t = -\psi_{xx} + V(x)\,\psi$$

$$\psi_t = -i\,H\,\psi$$

$\psi = 0$ at $x \in \partial D$

Hermitian operator (the energy)



$$\Psi(x,t) = \sum_n c_n \Psi_n(x)\, e^{-iE_n t}$$

$|\Psi(x,t)|^2 =$ the probability density for the particle to be at the point x at time t

$$H\Psi_n = E_n \Psi_n$$

energy eigenstates

Time-dependent Schrödinger equation:

$$\psi_t = -i H(\varepsilon t)\,\psi$$

$$\varepsilon \ll E_{n+1} - E_n \quad \forall n$$

$$\Psi(x,t) = \sum_n c_n(t)\,\Psi_n(x;\theta)\, e^{-i\int E_n(\theta)\,dt}$$

$$\theta = \varepsilon t$$

$$H(\theta)\,\Psi_n(\theta) = E_n(\theta)\,\Psi_n(\theta)$$

frozen energy eigenstates

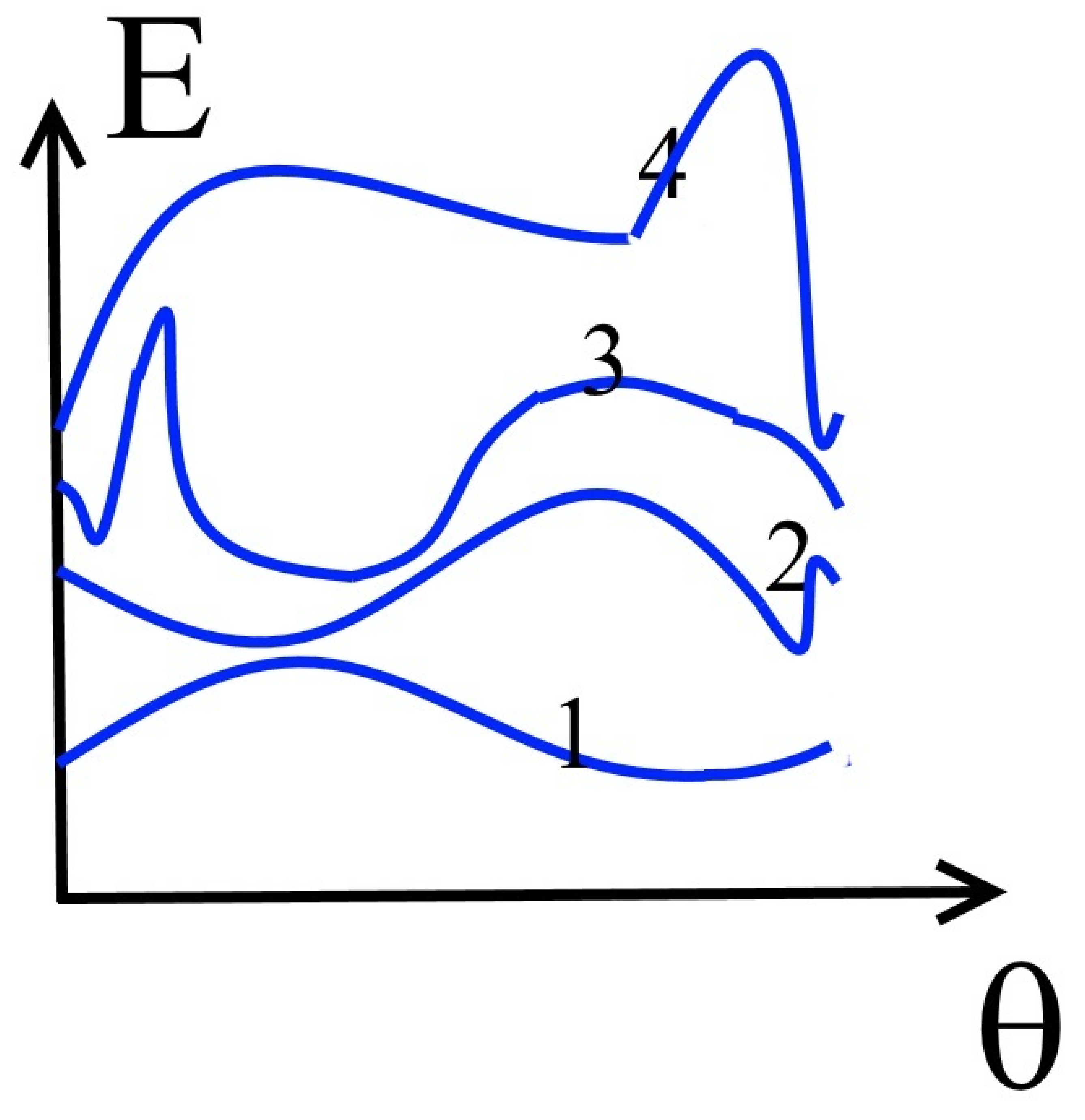
Adiabatic Theorem: $|c_n(t)|^2 \approx \text{const}$ for $t \gg \varepsilon^{-1}$

Proof: $$\frac{d}{dt}(c_n c_n^*) \approx \varepsilon\, e^{i(E_{n\pm1} - E_n)\,t}$$

If the system is close to a frozen energy eigenstate, it remains close to a frozen energy eigenstate for a long time (if the energy levels do not cross)

Avoided Crossing theorem:

for a typical one-parameter family of Hermitian operators $H(\theta)$ all eigenvalues are simple for all θ



Proof: $\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \sim \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \Rightarrow a=c=E,\ b=0 \Rightarrow$

$\Rightarrow$ 1-parameter family is not enough

For a typical periodic $H(\theta)$, the system described by

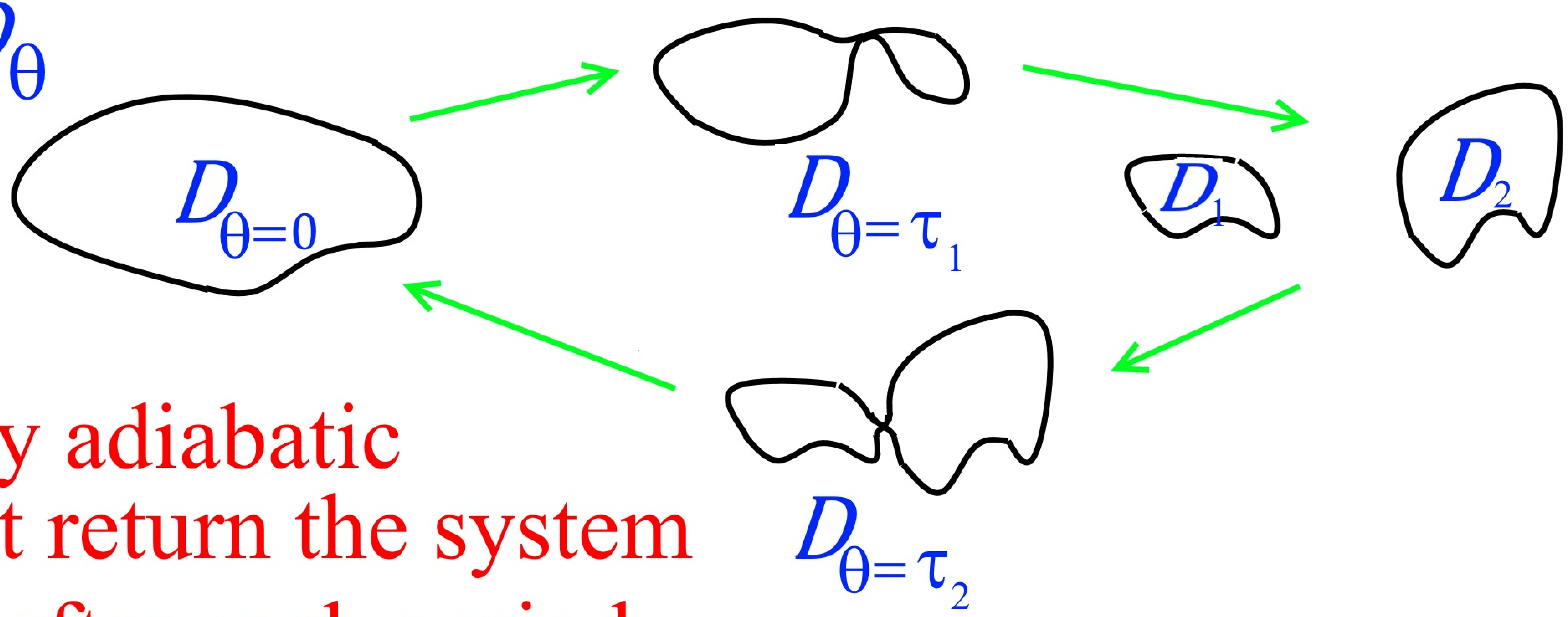
$\psi_t = -iH(\varepsilon t)\psi$ starting at an energy eigenstate

returns close to its initial state for a large number of periods

Schrödinger equation in a periodically separated domain:

$$i\,\psi_t = -\psi_{xx} + V(x,\theta)\,\psi$$

$$\psi = 0 \quad \text{at} \quad x \in \partial D_\theta$$



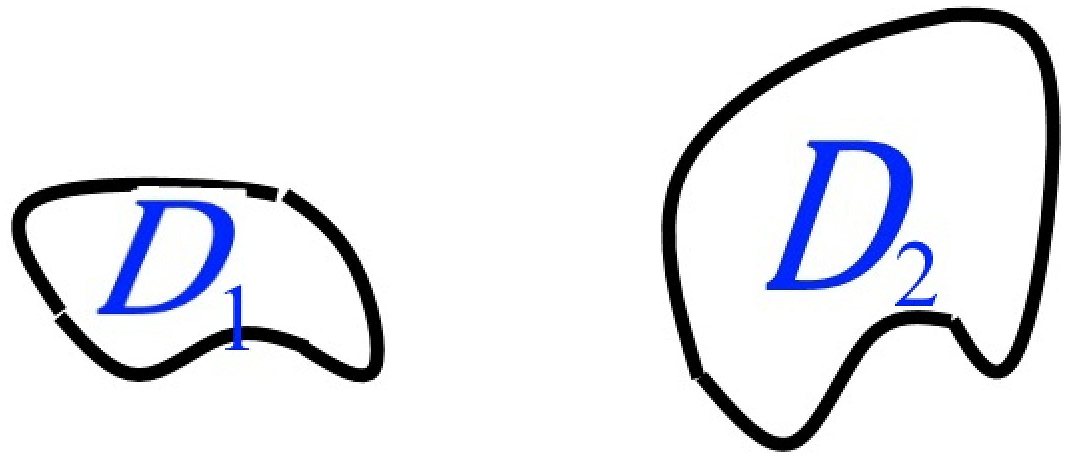
Typically, perfectly adiabatic evolution does not return the system to the initial state after each period.

Instead, the energy, typically, grows exponentially with time

Eigenfunctions between the separation and reconnection moments

$$E\psi = -\psi_{xx} + V(x,\theta)\psi \qquad \tau_1 \leqslant \theta \leqslant \tau_2$$

$\psi = 0$ at $x \in \partial D_1$ and $x \in \partial D_2$



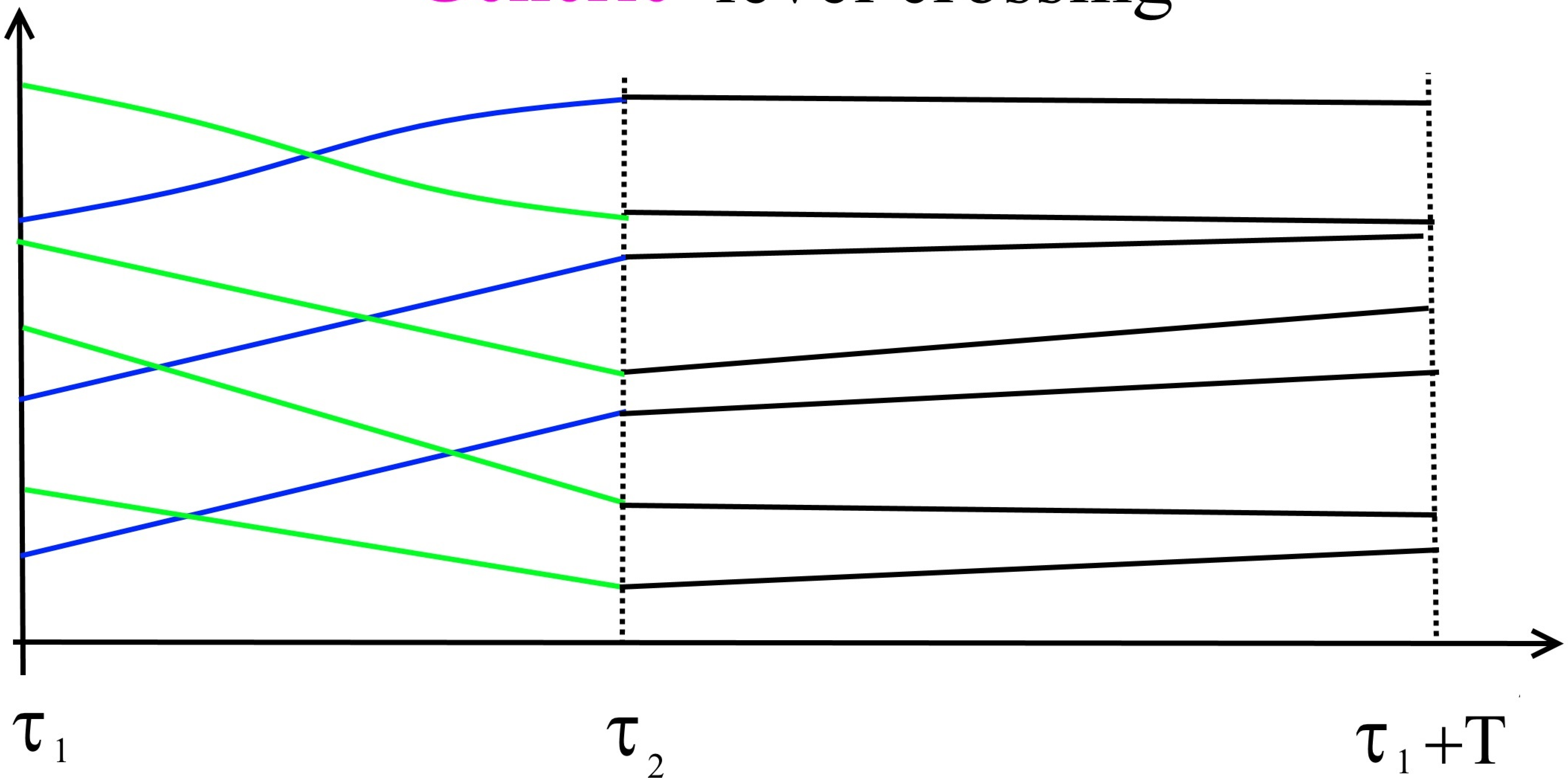
$\psi = 0$ for all $x \in D_1$ or all $x \in D_2$

2 groups of eigenfunctions and eigenvalues:

ψ_n^+, E_n^+ : particle in D_1

ψ_m^-, E_m^- : particle in D_2

Generic level crossing

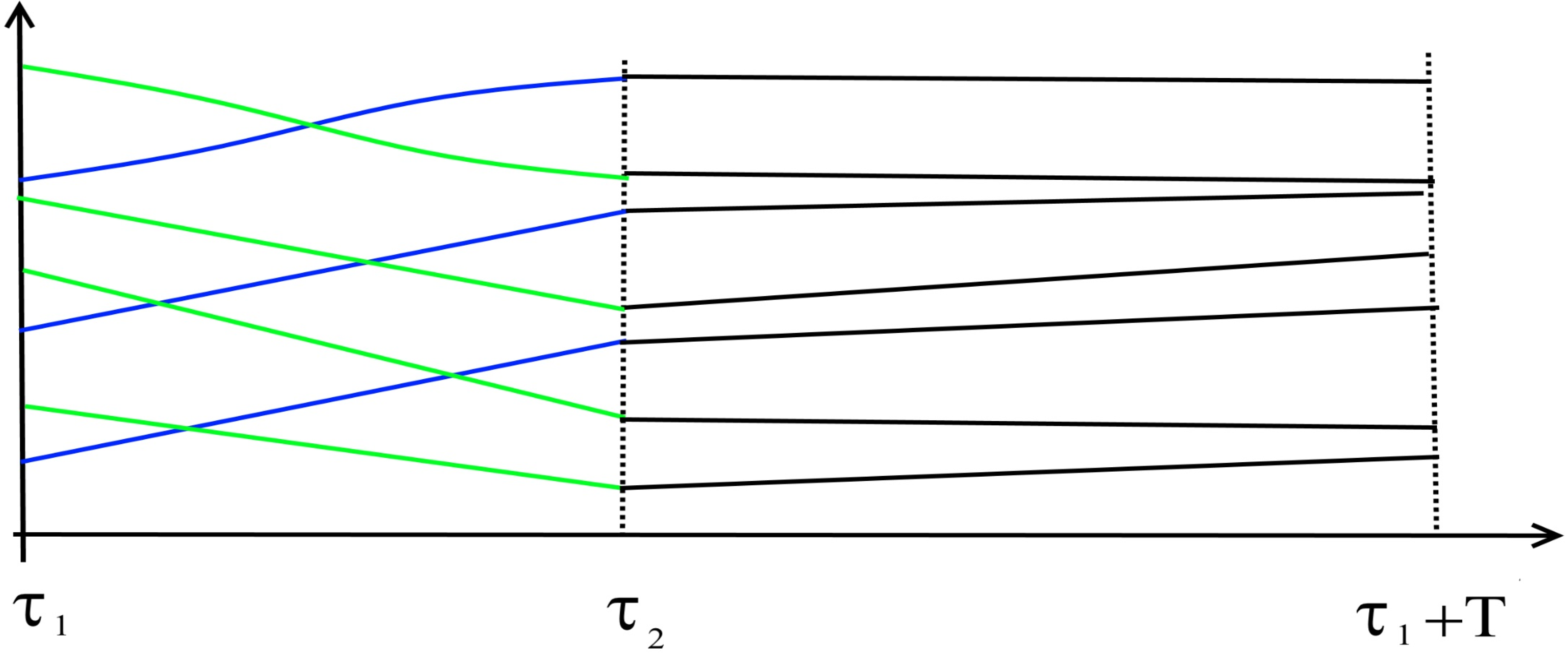


$\sigma_j(k) = 1$ if the eigenstate ψ_k belongs to group I at $\tau = \tau_j$

$\sigma_j(k) = -1$ if ψ_k belongs to group II at $\tau = \tau_j$

$(j = 1,2)$

$S_j(k) := \sigma_j(1) + \cdots + \sigma_j(k)$


τ_1
τ_2
$\tau_1 + T$


$\sigma_1(k)=1$
Group I state ψ_m
$S_1(k)= m - (k-m)$
$\sigma_2(\bar{k})=1$
Energy level number
k
$\sigma_1(k)S_1(k)+k= \sigma_1(k)S_2(\bar{k})+\bar{k}$
new level number
$\bar{k}$
$\sigma_1(k)=-1$
$S_2(k)=-n+(k-n)$
Group II state ψ_n
$\sigma_2(\bar{k})=-1$

The system from the energy level k at the separation occurs at the energy level $\bar{k}$ after the reconnection

The map $k \longrightarrow \bar{k}$ is 1-to-1 and is completely determined by the rule:

$$\sigma_2(\bar{k}) = \sigma_1(k) \qquad \sigma_1(k)S_1(k)+k = \sigma_1(k)S_2(\bar{k})+\bar{k}$$

The orbits of such maps are either periodic or tend to infinity in both directions

A generic orbit is unbounded and tends to infinity

exponentially

Exponential acceleration

Model assumptions: $\sigma_1(k)$ and $\sigma_2(k)$ are sequences of independent random variables

$\Pr(\sigma_1(k) = 1) = \beta$ $\Pr(\sigma_2(k) = 1) = \gamma$

$\beta \neq \gamma$

$S_1(k) \sim (2\beta - 1)k$ and $S_2(\bar{k}) \sim (2\gamma - 1)\bar{k}$

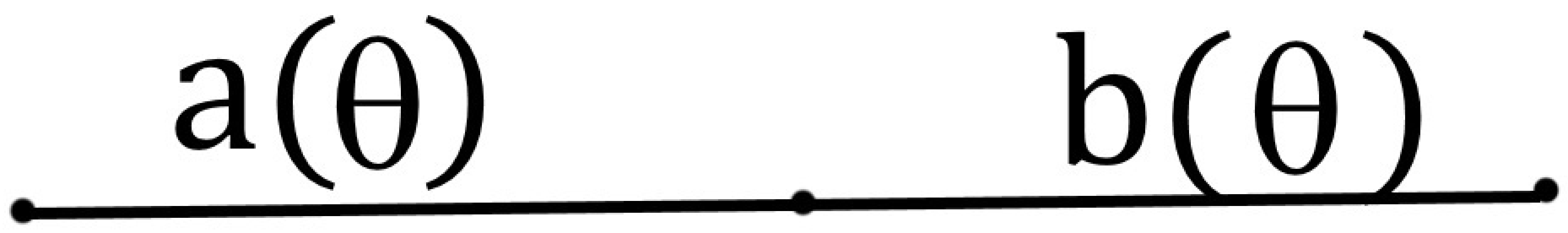
$\bar{k} + \sigma_1(k) S_2(\bar{k}) = k + \sigma_1(k) S_1(k)$

$\bar{k} \sim \frac{\beta}{\gamma} k$ with probability β

$\bar{k} \sim \frac{1-\beta}{1-\gamma} k$ with probability $1 - \beta$

$$E(\ln \bar{k} - \ln k) \sim \beta \ln \frac{\beta}{\gamma} + (1 - \beta) \ln \frac{1 - \beta}{1 - \gamma} > 0$$

Exactly solvable models: Quantum graphs with periodically broken links



$$E_n^- = \frac{\pi^2}{a^2} n^2 \qquad E_m^+ = \frac{\pi^2}{b^2} m^2$$

If, for a part of the adiabatic oscillation period, the frozen system has an additional quantum integral, which gets destroyed for the other part of the period, then the operator of the adiabatic evolution over the period is non-trivial. Generically, it has an infinite Jordan block.

Generically, the energy in such systems grows exponentially with time

1. Other quanum integrals?

2. Other equations with periodically divided domains?

3. Optics?

$$-i A_z = A_{xx} + A_{yy} + n(x,z) A$$