Generalized kernels of polygons under rotation*

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Abstract

Given a set $\mathcal{O}$ of k orientations in the plane, two points inside a simple polygon P *$\mathcal{O}$-see each other* if there is an $\mathcal{O}$-staircase contained in P that connects them. The $\mathcal{O}$-kernel of P is the subset of points which $\mathcal{O}$-see all the other points in P. This work initiates the study of the computation and maintenance of the $\mathcal{O}$-Kernel of a polygon P as we rotate the set $\mathcal{O}$ by an angle θ, denoted $\mathcal{O}$-Kernel$_\theta(P)$. In particular, we design efficient algorithms for (i) computing and maintaining $\{0^\circ\}$-Kernel$_\theta(P)$ while θ varies in $[-\frac{\pi}{2}, \frac{\pi}{2})$, obtaining the angular intervals where the $\{0^\circ\}$-Kernel$_\theta(P)$ is not empty and (ii) for orthogonal polygons P, computing the orientation $\theta \in [0, \frac{\pi}{2})$ such that the area and/or the perimeter of the $\{0^\circ, 90^\circ\}$-Kernel$_\theta(P)$ are maximum or minimum. These results extend previous works by Gewali, Palios, Rawlins, Schuierer, and Wood.

1 Introduction

The problem of computing the kernel of a polygon is a well-known visibility problem in computational geometry [2, 3, 6], closely related to the problem of guarding a polygon and to the motion problem of a robot inside a polygon with the restriction that the robot path must be *monotone* in some predefined set of orientations [5, 7, 8]. The present contribution goes a step further in the latter setting, by allowing that set of predefined orientations to rotate.

A curve $\mathcal{C}$ is *$0°$-convex* if its intersection with any $0°$-line (parallel to the x-axis) is connected or, equivalently, if the curve $\mathcal{C}$ is y-monotone. Extending this definition, a curve $\mathcal{C}$ is *α-convex* if the intersection of $\mathcal{C}$ with any line forming angle α with the x-axis is connected or, equivalently, if the curve $\mathcal{C}$ is monotone with respect to the direction perpendicular to such a line. Let us now consider a set $\mathcal{O}$ of k orientations in the plane, each of them given by an oriented line ℓ_i through the origin of the coordinate system and forming a counterclockwise angle α_i with the positive x-axis, so that $\mathcal{O} = \{\alpha_1, \ldots, \alpha_k\}$. Then, a curve is *$\mathcal{O}$-convex* if it is α_i-convex for all i, $1 \le i \le k$. Notice that the orientations in $\mathcal{O}$ are between 0° and 180°. From now on, an $\mathcal{O}$-convex curve will be called an *$\mathcal{O}$-staircase*.

▶ **Definition 1.1.** Let p and q be points inside a simple polygon P. We say that p and q *$\mathcal{O}$-see each other* or that they are *$\mathcal{O}$-visible from each other* if there is an $\mathcal{O}$-staircase contained in P that connects p and q. The *$\mathcal{O}$-Kernel* of P, denoted $\mathcal{O}$-Kernel(P), is the subset of points

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in P which $\mathcal{O}$-see all the other points in P. We denote by $\mathcal{O}$-Kernel$_\theta(P)$ the $\mathcal{O}$-kernel when the set $\mathcal{O}$ is rotated by an angle θ.

Schuierer, Rawlins, and Wood [7] defined the restricted orientation visibility or *$\mathcal{O}$-visibility* in a simple polygon P with n vertices, giving an algorithm to compute the $\mathcal{O}$-Kernel(P) in time $O(k + n \log k)$ with $O(k \log k)$ preprocessing time to sort the set $\mathcal{O}$ of k orientations. In order to do so, they used the following observation:

▶ **Observation 1.2.** *For any simple polygon P, the $\mathcal{O}$-Kernel(P) is $\mathcal{O}$-convex and connected. Furthermore, $\mathcal{O}$-Kernel$(P) = \bigcap_{\alpha_i \in \mathcal{O}} \alpha_i$-Kernel$(P)$.*

The computation of the $\mathcal{O}$-Kernel has been considered by Gewali [1] as well, who described an $O(n)$-time algorithm for orthogonal polygons without holes and an $O(n^2)$-time algorithm for orthogonal polygons with holes. The problem is a special case of the one considered by Schuierer and Wood [9] whose work implies an $O(n)$-time algorithm for orthogonal polygons without holes and an $O(n \log n + m^2)$-time algorithm for orthogonal polygons with $m \geq 1$ holes. More recently, Palios [5] gave an output-sensitive algorithm for computing the $\mathcal{O}$-Kernel of an n-vertex orthogonal polygon P with m holes, for $\mathcal{O} = \{0^\circ, 90^\circ\}$. This algorithm runs in $O(n + m \log m + \ell)$ time where $\ell \in O(1 + m^2)$ is the number of connected components of the $\{0^\circ, 90^\circ\}$-Kernel(P). Additionally, a modified version of this algorithm computes the number ℓ of connected components of the $\{0^\circ, 90^\circ\}$-Kernel in $O(n + m \log m)$ time.

2 The rotated $\{0^\circ\}$-Kernel$_\theta(P)$

In this section, for a given n-vertex simple polygon P, we deal with the rotation by $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$ of the $\mathcal{O}$-Kernel(P) in the particular case of $\mathcal{O} = \{0^\circ\}$. For the steady case $\theta = 0$, when the kernel is composed by the points which see every point in P via a y-monotone curve, Schuierer, Rawlins, and Wood [7] presented the following results.

▶ **Definition 2.1.** A reflex vertex p_i in P where p_{i-1} and p_{i+1} are both below (respectively, above) p_i is a *reflex maximum* (respectively, *reflex minimum*). A horizontal edge with two reflex vertices may also form a reflex maximum or minimum.

Let $C(P)$ be the (infinite) strip defined by the horizontal lines h_N and h_S passing through the lowest reflex minimum, p_N, and the highest reflex maximum, p_S.

▶ **Lemma 2.2** ([7]). *The $\{0^\circ\}$-Kernel(P) is the intersection $C(P) \cap P$.*

▶ **Observation 2.3.** *Notice that P may not have a reflex maximum, and in that case we have to take as the highest reflex maximum the lowest vertex of P. Analogously, if P has no reflex minimum, then we take as the lowest reflex minimum the highest vertex of P. Notice also that a horizontal edge of P may also form a reflex maximum or minimum. If P is a convex polygon then the $\{0^\circ\}$-Kernel(P) is the whole P.*

▶ **Corollary 2.4** ([7]). *The $\{0^\circ\}$-Kernel(P) can be computed in $O(n)$ time.*

It is clear that there are neither reflex minima nor reflex maxima within $C(P)$. Moreover, the lines h_N and h_S contain the segments of the *north* and *south* boundary of the $\{0^\circ\}$-Kernel(P). See Figure 1, left.

Let c^l and c^r denote the left and the right polygonal chains defined by the respective left and right parts of the boundary of P inside $C(P)$. Because of Lemma 2.2, both chains are 0°-convex curves, i.e., y-monotone chains.

▶ **Corollary 2.5.** *The area and perimeter of* $\{0^\circ\}$-Kernel(P) *can be computed in* $O(n)$ *time.*

Proof. To compute the area of the $\{0^\circ\}$-Kernel$(P) = C(P) \cap P$ we proceed as follows. The lines h_N and h_S contain the segments of the north and south boundary of the $\{0^\circ\}$-Kernel(P). The area can be computed by decomposing it into (finite) horizontal strips, which in fact are trapezoids defined by the edges of P inside $C(P)$ in such a way that correspond to consecutive vertices from the merging of the sorted list of the vertices of either c^l or c^r. Each strip is computed in constant time, and therefore, the area of $\{0^\circ\}$-Kernel$(P) = C(P) \cap P$ can be computed in $O(|c^l| + |c^r|)$ time.

Computing the perimeter is simpler, because we only need to add the lengths of both chains c^l and c^r and the lengths of the sides of the north and south boundary of the $\{0^\circ\}$-Kernel(P); this requires computing the intersection of the lines h_N and h_S with the boundary of P and can be done in $O(|c^l| + |c^r|)$ time. ◀

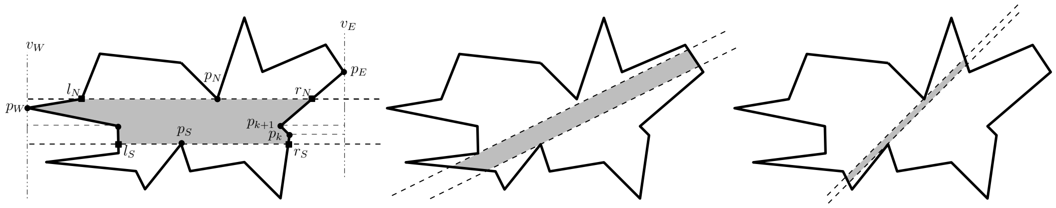

Figure 1 A rotating $\{0°\}$-Kernel$_\theta(P)$ for $\theta = 0$ (left), $\theta = \frac{\pi}{8}$ (middle), and $\theta = \frac{\pi}{4}$ (right).

Next, we will compute the $\{0^\circ\}$-Kernel$_\theta(P)$ as θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. To do that, first we need to maintain the strip boundary $C_\theta(P)$ during the variation of θ, i.e., the equations of the two horizontal lines $h_N(\theta)$ and $h_S(\theta)$ which contain the current horizontal sides of $\{0^\circ\}$-Kernel$_\theta(P)$, in such a way that for a given value of θ the boundary of $C_\theta(P)$ can be computed from these equations. Second, we also need to maintain the set of vertices of the left and right chains c^l_θ and c^r_θ of $\{0^\circ\}$-Kernel$_\theta(P)$.

First, we observe that Definition 2.1 of a reflex maximum and a reflex minimum, with respect to the horizontal orientation, extends easily for any given orientation $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$. Then, for every reflex vertex $p_i \in P$ we compute in constant time the angular interval $(\theta^i_1, \theta^i_2) \subset [-\frac{\pi}{2}, \frac{\pi}{2})$ of the orientations such that $p_i \in P$ is a candidate reflex maximum or reflex minimum. Finally, in constant time we compute the corresponding *slope intervals* $(m^i_1, m^i_2) \subset (-\infty, \infty)$ for the lines with these orientations. Using these to sweep in the dual plane [4], we get the following result, where $\alpha(n)$ is the extremely slowly-growing inverse of Ackermann's function and $O(n\,\alpha(n))$ comes from the complexity of the upper and lower envelopes of a set of n straight line segments in the plane:

▶ **Theorem 2.6.** *For an n-vertex simple polygon P, there are $O(n\,\alpha(n))$ angular intervals $(\theta_1, \theta_2) \subset [-\frac{\pi}{2}, \frac{\pi}{2})$ such that* $\{0^\circ\}$-Kernel$_\theta(P) \neq \emptyset$ *for all the values of $\theta \in (\theta_1, \theta_2)$, and the set of such intervals can be computed in $O(n \log n)$ time.*

▶ **Observation 2.7.** *To maintain the polygonal chains c^l_θ and c^r_θ of* $\{0^\circ\}$-Kernel$_\theta(P)$ *we compute the intersections of the lines $h_N(\theta)$ and $h_S(\theta)$ with the boundary of P, maintaining the information of the first and the last vertices of c^l_θ and c^r_θ in the current interval (θ_1, θ_2).*

3 The rotated $\{0^\circ, 90^\circ\}$-Kernel$_\theta(P)$ of simple orthogonal polygons

We now extend our study to $\mathcal{O} = \{0^\circ, 90^\circ\}$ for the particular case of orthogonal polygons, where it suffices to consider $\theta \in [0, \frac{\pi}{2})$.

Each edge of an orthogonal polygon is a N-*edge*, S-*edge*, E-*edge*, or W-*edge* if it bounds the polygon from the north, south, east, or west, respectively. For $D \in \{\mathrm{N}, \mathrm{S}, \mathrm{E}, \mathrm{W}\}$, a D-edge is a D-*dent* (resp., D-*extremity*) if both of its endpoints are reflex (resp., convex) vertices of the polygon. Next, a NE-*staircase* is a sequence of alternating N- and E-edges and similarly we can define the NW-*staircase*, SE-*staircase*, and SW-*staircase*; clearly, each such staircase is both x- and y-monotone. Finally, depending on the type of incident edges, a *reflex* vertex of the given orthogonal polygon P is called a NE-*reflex vertex* if it is incident to a N-edge and an E-edge, and analogously for NW-, SE- and SW-*reflex vertices*. See Fig. 2(a).

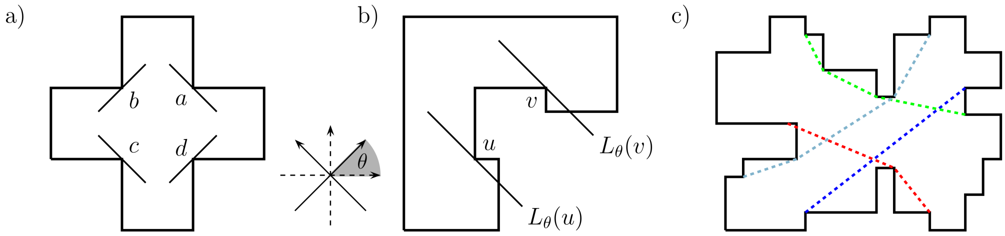

Figure 2 (a) Clipping lines through the reflex vertices of the polygon, where a, b, c, d are respectively a NE-, NW-, SW-, and SE-reflex vertex. (b) Illustration for Lemma 3.2. (c) An orthogonal polygon belonging to the family $\mathcal{Q}$.

The size of the kernel. For $\theta = 0$, in accordance with Observation 1.2 and Lemma 2.2, the $\{0^\circ, 90^\circ\}$-Kernel$_0(P)$ equals the intersection of P with the four *closed* halfplanes defined below the lowermost N-dent, above the topmost S-dent, right of the rightmost W-dent, and left of the leftmost E-dent, if such dents exist; thus, the $\{0^\circ, 90^\circ\}$-Kernel$_0(P)$ can be computed in $O(n)$ time. For $\theta \in (0, \frac{\pi}{2})$, the $\{0^\circ, 90^\circ\}$-Kernel$_\theta(P)$ can be obtained by clipping P about appropriate lines; it turns out that *every* reflex vertex contributes such a clipping line. In particular, each NW- or SE-reflex vertex contributes a clipping line parallel to 0°_θ whereas each NE- or SW-reflex vertex contributes a clipping line parallel to 90°_θ, where α_θ denotes the rotation of the orientation α by the angle θ; see Fig. 2(a). Thus, if there exist reflex vertices of all four types, the $\{0^\circ, 90^\circ\}$-Kernel$_\theta(P)$ has two pairs of sides, with one pair parallel to each of the rotated orientations. Moreover, an extremity may contribute an edge to the kernel; however, we can prove that only one extremity from each of the four kinds may do so [4]. On the other hand, if the polygon has no NE-reflex vertices then there is a unique N-extremity and a unique E-extremity, which share a common endpoint, and then the $\{0^\circ, 90^\circ\}$-Kernel$_\theta(P)$ has no additional edges in such a case. Similar results hold in the cases of absence of NW-reflex, SW-reflex, or SE-reflex vertices. Therefore, we obtain:

▶ **Lemma 3.1.** *The rotated $\{0°, 90°\}$-Kernel$_\theta(P)$ has at most eight edges.*

In order to compute the $\{0^\circ, 90^\circ\}$-Kernel$_\theta(P)$ for each value of θ, we need to collect the sets of the four types of reflex vertices of P and compute the convex hull of each of them, which will give us the corresponding minimum or maximum. We can show the following ($v.x$ and $v.y$ denote the x- and y-coordinates of a vertex v, respectively):

▶ **Lemma 3.2.** *Let P be an orthogonal polygon. If u and v are a* NE*-reflex and a* SW*-reflex vertex of P, respectively, such that $v.x \geq u.x$ and $v.y \geq u.y$ (see Fig.* 2(b)*), then for each $\theta \in (0, \frac{\pi}{2})$, the $\{0^\circ, 90^\circ\}$-Kernel$_\theta(P)$ is empty. Similarly, if u', v' are a* NW*-reflex and a* SE*-reflex vertex of P, respectively, such that $v'.x \leq u'.x$ and $v'.y \geq u'.y$, then for each $\theta \in (0, \frac{\pi}{2})$, the $\{0^\circ, 90^\circ\}$-Kernel$_\theta(P)$ is empty.*

Proof. For each $\theta \in (0, \frac{\pi}{2})$, let $L_\theta(u), L_\theta(v)$ be the lines forming angle $90^\circ + \theta$ with the positive x-axis that are tangent at u, v, respectively (see Fig. 2(b)). Clearly, $L_\theta(u)$ is below and to the left of $L_\theta(v)$. This leads to a contradiction since the kernel can neither be higher than $L_\theta(u)$ nor lower than $L_\theta(v)$. ◀

Next, let $\mathcal{Q}$ be the family of simple orthogonal polygons whose boundary (in ccw traversal) consists of the concatenation of (i) a y-monotone chain from the lowermost E-extremity to the topmost E-extremity, (ii) a NE-staircase to the rightmost N-extremity, (iii) an x-monotone chain to the leftmost N-extremity, (iv) a NW-staircase to the topmost W-extremity, (v) a y-monotone chain to the lowermost W-extremity, (vi) a SW-staircase to the leftmost S-extremity, (vii) an x-monotone chain to the rightmost S-extremity, and (viii) a SE-staircase to the lowermost E-extremity. See Fig. 2(c). Note that each pair of extremities and the monotone chain between them may degenerate into a single extremity, whereas each staircase may degenerate into a 2-edge chain.

▶ **Corollary 3.3.** *If $P \notin \mathcal{Q}$, then the $\{0^\circ, 90^\circ\}$-Kernel$_\theta(P)$ is empty for each $\theta \in (0, \frac{\pi}{2})$.*

Proof. Let $\mathcal{E}$ (resp., $\mathcal{N}$) be the subset of E-extremities (resp., N-extremities) such that $e \in \mathcal{E}$ (resp., $e \in \mathcal{N}$) iff for every point p of $e \in \mathcal{E}$ (resp., $e \in \mathcal{N}$), the rightward horizontal ray (resp., upward vertical ray) from p does not intersect the interior of P; clearly, $\mathcal{E} \neq \emptyset$ and $\mathcal{N} \neq \emptyset$.

If the ccw chain of edges from the topmost E-extremity e in $\mathcal{E}$ to the rightmost N-extremity e' in $\mathcal{N}$ contains a S-edge, and let e'' be the first encountered S-edge from e to e', then the rightmost edge in the ccw boundary chain from e'' to e' is an E-extremity in $\mathcal{E}$ higher than e, whereas if it contains an W-edge then the edge preceding the first-encountered W-edge in the ccw boundary chain from e to e' is a N-extremity in $\mathcal{N}$ to the right of e'; a contradiction. Hence, the ccw boundary chain from e to e' is a NE-staircase. Next, we show that if the ccw chain of edges from the lowermost E-extremity e_1 in $\mathcal{E}$ to the topmost E-extremity e_2 in $\mathcal{E}$ is not y-monotone, then the $\{0^\circ, 90^\circ\}$-Kernel$_\theta(P)$ is empty for each $\theta \in (0, \frac{\pi}{2})$. For contradiction, suppose that this chain is not y-monotone. Then, it contains an W-edge; let e be the first encountered such edge. Two cases are possible; see Figure 3.

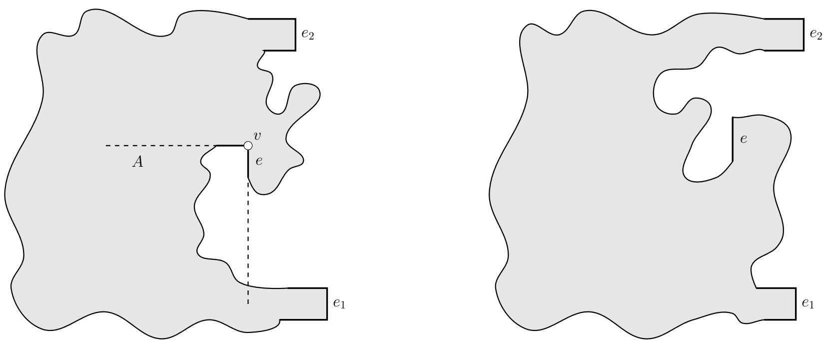

Figure 3 The two possible cases for the proof of Corollary 3.3.

In the case shown on the left in Figure 3, the edge preceding e is a S-edge and their common endpoint v is a SW-reflex vertex. Let A be the bottom-left quadrant defined by the lines supporting the edges incident on v and let ρ be the part of the boundary chain from e_1 to e that belongs to A. Then, the bottom vertex of the leftmost edge in ρ is a NE-reflex vertex which is to the left and below v, and Lemma 3.2 implies that the $\{0^\circ, 90^\circ\}$-Kernel$_\theta(P)$ is empty for each $\theta \in (0, \frac{\pi}{2})$, as desired. The case shown on the right in Figure 3 is a top-down mirror image of the case shown on the left; hence, the same result holds also for this case. ◀

Algorithm. Corollary 3.3 shows that the $\{0^\circ, 90^\circ\}$-Kernel$_\theta$ of orthogonal polygons not in $\mathcal{Q}$ is empty for all $\theta \in (0, \frac{\pi}{2})$. For those in $\mathcal{Q}$, their particular form allows the efficient computation of the kernel. In particular, for any polygon $P \in \mathcal{Q}$, a single traversal of the vertices of P allows us to compute the maxima of the SW-reflex vertices of P in linear time. Indeed, the SW-reflex vertices of such a polygon P belong to the ccw boundary chain from the topmost W-extremity to the rightmost S-extremity; then, both the maxima in the y-monotone chain from the topmost W-extremity to the leftmost S-extremity and the maxima in the x-monotone chain from the leftmost S-extremity to the rightmost S-extremity can be computed in linear time, and the two maxima sequences can be merged in linear time as well. Then, the cw part of the boundary of their convex hull from its topmost to its rightmost vertex (the red dashed line in Fig. 2(c)), which is the useful one, can also be computed in linear time by using the iterative step of the Graham scan algorithm. In a similar fashion, we can compute in linear time the useful boundary parts of the convex hulls of the other three types of reflex vertices; see the dashed lines in Fig. 2(c).

Since the $\{0°, 90°\}$-Kernel$_\theta(P)$ for each angle $\theta \in [0, \frac{\pi}{2})$ is mainly determined by the, at most, 4 reflex vertices through which clipping lines are passing (Lemma 3.1), the algorithm needs to maintain these vertices. Therefore, for each edge of each of the at most 4 convex hulls, we compute the corresponding angle $\theta \in [0, \frac{\pi}{2})$ and we associate it with the edge's endpoints, and these angles are sorted by merging the 4 ordered lists of angles. Then, for each pair of consecutive angles $\theta_{prev}, \theta_{cur}$, we have the reflex vertices through which the clipping lines pass for any $\theta \in [\theta_{prev}, \theta_{cur})$, and we compute the at most 8 vertices of the kernel parameterized on θ. Next, we maximize the kernel's area/perimeter for $\theta \in [\theta_{prev}, \theta_{cur})$, and if needed, we update the overall maximum and store the corresponding angle. Thus, we have:

▶ **Theorem 3.4.** *Given a simple orthogonal polygon P on n vertices, the value of the angle $\theta \in [0, \frac{\pi}{2})$ such that the $\{0^\circ, 90^\circ\}$-Kernel$_\theta(P)$ has maximum (or minimum) area/perimeter can be computed in $O(n)$ time and space.*

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