Lower bounds for coloring of the plane

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Abstract

Let $G_{[1,b]}$ be the graph with the set of vertices $\mathbb{R}^2$ and adjacency between points at distance in the set $[1,b]$. We obtain new lower bounds for $\chi(G_{[1,b]})$ for certain values of b. Combined with known upper bounds, this result gives two intervals of values of b for which we exactly determine $\chi(G_{[1,b]})$ to be 7 and 9, respectively. The first interval contains and substantially enlarges the only known set of values of b with determined $\chi(G_{[1,b]})$ coming from the work of Exoo (2004).

1 Introduction

1.1 Background

Probably the most known and challenging question of geometric graph theory is the Hadwiger-Nelson problem. It asks for the minimal number of colors in a coloring of the Euclidean plane $\mathbb{R}^2$ with a restriction that any two points at distance 1 obtain distinct colors. In other words: what is the chromatic number of the unit distance graph of $\mathbb{R}^2$ (a graph defined on the set of vertices $\mathbb{R}^2$ and with edges between vertices at distance 1)? The problem was first posed by Edward Nelson in 1950 and made known to the mathematical society by Hugo Hadwiger. As soon as in 1950's, Nelson and John Isbell were first to prove that the answer is at least 4 and at most 7, respectively. Unfortunately, after several decades, these classic bounds are still the best known in general. Although far from being solved, Hadwiger-Nelson problem inspired a vast number of challenging questions, interesting results and applications in the intersection of combinatorics and geometry. For more information on the history of the problem and selected related problems, we refer to the book of Soifer [9].

One of possible directions in order to obtain more understanding of such problems is to consider a more general set of restricting distances. Let $G_{[a,b]}$ denote the graph with the set of vertices $\mathbb{R}^2$ and two vertices adjacent if they are at distance from the interval $[a,b]$. However, by scaling, we can assume that $a = 1$. In this paper, we will consider these graphs.

Some important results on coloring of such graphs were presented by Exoo [2] (using slightly different notation). In particular, he showed the following theorem.

▶ **Theorem 1.1** ([2]).
For $b \in (\sqrt{43}/5, \sqrt{7}/2] \approx (1.31149, 1.32287]$ *there holds* $\chi(G_{[1,b]}) = 7$.

To our knowledge, the interval given in Theorem 1.1 was the only known set of values of b such that $\chi(G_{[1,b]})$ was determined. We note that the real contribution of Theorem 1.1 lays in establishing the lower bound for $\chi(G_{[1,b]})$. The upper bound comes from the observation that the well known 7-coloring of $G_{[1,1]}$ based on hexagonal tiling is proper also for $G_{[1,b]}$ for any $b \leq \sqrt{7}/2$. Exoo provided also a small $\delta > 0$ such that for any $b > 1 + \delta$, it holds that $\chi(G_{[1,b]}) \geq 5$. Later, it was published in [3] that the latter statement can be strengthened to any $b > 1$. However, it appears that before the mentioned two papers, the last result was already surpassed by two independent works. In a series of papers by Brown, Dunfield, Perry [6–8], among other results, the authors gave an elegant proof by Dunfield that for any $b > 1$ we have $\chi(G_{[1,b]}) \geq 6$. The proof is based on a result by Woodall (incorrect proof [12]) and Townsend (correct proof based on similar idea, see [10, 11]). Without giving the precise

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statement, the Woodall-Townsend theorem can be expressed in the following way: if the unit distance graph of the plane is colored with the condition that color classes are defined with Jordan curves, then at least 6 colors are necessary. The key ingredient of the proof of the Woodall-Townsend theorem, very roughly speaking, is to find a point in the plane, which has at least 3 colors in any ε-neighborhood. A related idea was used by Currie and Eggleton in their manuscript [5], where they independently prove the same result as Dunfield. Although the manuscript was not properly published, it was already mentioned by Currie in his other paper published in 1992 [1]. Currie and Eggelton consider a coloring of $G_{[1,b]}$ and then for $\varepsilon \in \big(0, (b-1)/2\big)$ find a point x for which the closed ε-ball centered at x contains at least 3 colors. Then they prove that the annulus $\{p \in \mathbb{R}^2 : 1+\varepsilon \leq dist(p,x) \leq b-\varepsilon\}$ needs at least 3 colors and observe that it cannot use any of the colors from the closed ε-ball centered at x. This ends the proof.

1.2 Our approach

It seems that the idea of a point close to at least 3 colors was not exploited for larger values of b. In our work, we use this concept to provide new lower bounds for $\chi(G_{[1,b]})$ for certain values of $b > 1$. The approach consists of two steps. First, we use the mentioned fact that any coloring of $G_{[1,b]}$ for any $b > 1$ and any sufficiently small $\varepsilon > 0$ admits a closed ε-ball centered at some point x containing at least 3 colors. We give a new proof of this statement. As the second step, we consider the annulus $A_{b,\varepsilon}$ centered at x with the inner radius $1+\varepsilon$ and the outer radius $b-\varepsilon$. Clearly, none of at least 3 colors found in the closed ε-ball centered at x can be used in $A_{b,\varepsilon}$. If for some k we are able to prove that $A_{b,\varepsilon}$ itself requires at least k colors, then we obtain $\chi(G_{[1,b]}) \geq k+3$.

In order to show a lower bound for coloring of $G_{[1,b]}$ or a subset of $G_{[1,b]}$, one may try to construct a finite subset for which finite graph coloring techniques can be applied. For example Exoo, in order to prove the lower bound in Theorem 1.1, considered coloring of a finite part P of a carefully chosen regular triangular grid. Using computer aided calculations, he showed that for the specified range of b the subgraph of $G_{[1,b]}$ induced by P requires at least 7 colors. However, it is unlikely that his choice of parameters for the grid is optimal (in terms of range of b), as it is limited by the computer computational power. In order to show a lower bound for coloring of $A_{b,\varepsilon}$, we also construct a certain finite subset of it. Our analysis suggested that it is reasonable to consider sets created by taking a number of points regularly placed on a small number of circles of radius chosen between $1+\varepsilon$ and $b-\varepsilon$.

The benefit of this approach is that we reduce the search for finite configurations to a relatively small part of the plane. On the other hand, it is likely that for many values of b the chromatic number of the subgraph of $G_{[1,b]}$ induced by $A_{b,\varepsilon}$ plus 3 is strictly smaller than $\chi(G_{[1,b]})$. However, this plan proves itself to be effective in providing a new contribution, as we were able to determine $\chi(G_{[1,b]})$ for two intervals of values of b. In particular, we improve the important part of Theorem 1.1 in the meaning that we give a more general lower bound.

Our lower bounds on the number of colors for finite configurations are obtained by computer-based computations. We used a slightly modified standard mixed integer programming formulation of graph coloring and solved the models using IBM ILOG CPLEX solver (version 12.7.1).

2 The results

We start with a key lemma already proved in [5]. However, we give a new, shorter proof of this fact.

▶ **Lemma 2.1** ([5]; a new proof).
Let c be a proper coloring of $G_{[1,b]}$ for $b > 1$. Consider any $\varepsilon > 0$ satisfying $b - 1 > \varepsilon$. Then there exists a point x in $\mathbb{R}^2$ such that in the closed ε-ball centered in x there are at least 3 colors (with respect to c).

Proof. For a monochromatic set A colored with c_1 and $X \subseteq \mathbb{R}^2$ denote by $S(A, X)$ the set of all c_1-colored points in X that can be obtained from A by a sequence of c_1-colored points belonging to X with consecutive distances at most ε. For $S, X \subseteq \mathbb{R}^2$, let $H_\varepsilon(S, X) = \{p \in X : \exists_{s \in S}\ dist(p, s) \leq \varepsilon\}$. In the proof, we will use the following observation.

(∗) If S is a bounded, connected set and X contains the unbounded component of $\mathbb{R}^2 \setminus S$, then there exists a simple closed curve C in $H_\varepsilon(S, X) \setminus S$ so that all points from S are inside C.

For a simple closed curve C, let $In(C)$ be the bounded component of $\mathbb{R}^2 \setminus C$.

Suppose to the contrary to the thesis, that there is no such point x. Let $X_1 = \mathbb{R}^2$. Take any point $y \in \mathbb{R}^2$. Set $S_1 = S(\{y\}, X_1)$. Note that S_1 is a bounded set. Otherwise it would contain a sequence of points of the same color with consecutive distances less than ε and realizing arbitrarily large distance. Hence S_1 would contain a pair of points at distance in $(1, b)$, a contradiction. Since $H_{\varepsilon/2}(S_1, X_1)$ is bounded and connected, by $(*)$ we can find a simple closed curve C_1 in $H_\varepsilon(S_1, X_1) \setminus H_{\varepsilon/2}(S_1, X_1) = H_{\varepsilon/2}(H_{\varepsilon/2}(S_1, X_1), X_1) \setminus H_{\varepsilon/2}(S_1, X_1)$ so that all points from $H_{\varepsilon/2}(S_1, X_1)$ are inside C_1. Observe that all points of C_1 have the same color, as otherwise we would have a closed ε-ball with 3 colors. We continue the construction for $i > 1$ in the following way. We set $X_i = X_{i-1} \setminus In(C_{i-1})$, $S_i = S(C_{i-1}, X_i)$. Since $H_{\varepsilon/2}(S_i, X_i)$ is bounded and connected, by $(*)$ we can find a simple closed curve C_i in $H_\varepsilon(S_i, X_i) \setminus H_{\varepsilon/2}(S_i, X_i) = H_{\varepsilon/2}(H_{\varepsilon/2}(S_i, X_i), X_i) \setminus H_{\varepsilon/2}(S_i, X_i)$ so that all points from $H_{\varepsilon/2}(S_i, X_i)$ are inside C_i. Again, all points of C_i have the same color, as otherwise we would have a closed ε-ball with 3 colors.

We claim that $diam(C_i) - diam(C_{i-1}) \geq \varepsilon$ for $i > 1$. Consider two points y_1, y_2 that realize $diam(C_{i-1})$ and take the line ℓ containing y_1, y_2. Let y'_1, y'_2 be the points from ℓ satisfying $dist(y'_1, y_1) = \varepsilon/2$, $dist(y'_2, y_2) = \varepsilon/2$ and $dist(y'_1, y'_2) = dist(y_1, y_2) + \varepsilon$. Clearly, $y'_1, y'_2 \in H_{\varepsilon/2}(S_i, X_i) \subseteq In(C_i)$ and hence $diam(C_i) \geq dist(y'_1, y'_2)$, as claimed. Thus $diam(C_i) - diam(C_{i-1}) \geq \varepsilon$. Therefore for sufficiently large i we have $diam(C_i) > 1$ and there are two points in C_i at distance from $(1, b)$. On the other hand, all points C_i have the same color, which contradicts with the fact that c is a proper coloring of $G_{[1,b]}$. ◀

▶ **Theorem 2.2.** *The following inequalities hold:*

1. $\chi(G_{[1,b]}) \geq 7$ *for* $b > \sqrt{2 - 2\sin(\frac{18\pi}{325})} \approx 1.28599$
2. $\chi(G_{[1,b]}) \geq 8$ *for* $b > \sqrt{2 + 2\sin(\frac{\pi}{38})} \approx 1.47145$
3. $\chi(G_{[1,b]}) \geq 9$ *for* $b > \sqrt{2 + 2\sin(\frac{7\pi}{45})} \approx 1.71433$

Proof. Consider $b > 1$ and a proper coloring c of $G_{[1,b]}$ with $\chi(G_{[1,b]})$ colors, say $1, \ldots, \chi(G_{[1,b]})$. Fix $\varepsilon > 0$. Let x be a point such that in the closed ε-ball centered at x there are at least 3 colors with respect to c, say colors $1, 2, 3$. Without loss of generality we can assume that $x = (0, 0)$. Then no point in the annulus $A_{b,\varepsilon} = \{p \in \mathbb{R}^2 : 1 + \varepsilon \leq dist(p, \mathbf{0}) \leq b - \varepsilon\}$ can be colored with any of the colors $1, 2, 3$.

The general outline is that in each case for a fixed b, we will construct a finite subset of $A_{b,\varepsilon}$ (for sufficiently small ε) such that it needs at least k colors, for some k. In other words, we will find a subgraph of $G_{[1,b]}$ consisting of vertices from $A_{b,\varepsilon}$ forcing k colors. This will imply that $\chi(G_{[1,b]}) \geq 3 + k$. Denote by X^n_r the set consisting of n points evenly distributed on the circle with the center in $(0, 0)$ and radius r so that one point lays in the set $\{0\} \times (0, +\infty)$.

1. Assume that $b > \sqrt{2 - 2\sin(\frac{18\pi}{325})}$. Consider $Y_\varepsilon = X^{1300}_{1+\varepsilon} \cup X^{1300}_{b-\varepsilon} \subseteq A_{b,\varepsilon}$. It can be checked that for sufficiently small $\varepsilon > 0$, any proper coloring of the graph induced by Y_ε in $G_{[1,b]}$ uses at least 4 colors.
2. Assume that $b > \sqrt{2 + 2\sin(\frac{\pi}{38})}$. Consider $Y_\varepsilon = X^{190}_{1+\varepsilon} \cup X^{190}_{b-\varepsilon} \subseteq A_{b,\varepsilon}$. It can be checked that for sufficiently small $\varepsilon > 0$, any proper coloring of the graph induced by Y_ε in $G_{[1,b]}$ uses at least 5 colors.
3. Assume that $b > \sqrt{2 + 2\sin(\frac{7\pi}{45})}$. Consider $Y_\varepsilon = X^{180}_{1+\varepsilon} \cup X^{180}_{(1+b)/2} \cup X^{180}_{b-\varepsilon} \subseteq A_{b,\varepsilon}$. It can be checked that for sufficiently small $\varepsilon > 0$, any proper coloring of the graph induced by Y_ε in $G_{[1,b]}$ uses at least 6 colors.

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The exact right-hand side values in three inequalities on b in Theorem 2.2 are the optimal values for which the given finite configurations of points posses the desired chromatic properties. That is, in each case for any smaller value of b, the given set Y_ε can be colored with fewer colors than stated in the theorem. Nevertheless, we are far from claiming optimality of the given sets. In Theorem 2.2 we simply present the best constructions that we were able to find and verify. We expect that there exist sets of similar form which work for (possibly only slightly) smaller values of b.

By combining Theorem 2.2 with previously known bounds, we can obtain two intervals of values of b for which the chromatic number can be determined. Namely, let us use that Exoo [2] observed that $\chi(G_{[1,b]}) \leq 7$ for $b \leq \sqrt{7}/2$ and Ivanov [4] showed that $\chi(G_{[1,b]}) \leq 9$ for $b \leq \sqrt{3}$.

▶ **Corollary 2.3.** *For* $b \in \left(\sqrt{2 - 2\sin(\frac{18\pi}{325})}, \sqrt{7}/2\right] \approx (1.28599, 1.32287]$ *it holds* $\chi(G_{[1,b]}) = 7$.

▶ **Corollary 2.4.** *For* $b \in \left(\sqrt{2 + 2\sin(\frac{7\pi}{45})}, \sqrt{3}\right] \approx (1.71433, 1.73205]$ *it holds* $\chi(G_{[1,b]}) = 9$.

Note that the first interval contains and substantially enlarges the interval obtained by Exoo in Theorem 1.1. Moreover, the second interval was not known at all.

3 Conclusions

We note that our method combines a theoretical reasoning of continuous nature and constructions of finite sets for which the coloring properties are checked by computer. Therefore, the approach differs from the previously used in the literature. Similar constructions for larger values of b and larger number of colors are in preparation. However, it seems that the method should work better for relatively small values of b (and hence small number of colors), as in this case the 3 colors reserved by the ε-ball make a greater difference.

One may observe that we do not have any interval with the chromatic number determined to 8 colors. The reason is that we do not have a good 8-coloring of the plane. That is, an 8-coloring of $G_{[1,b]}$ that would work for b substantially larger than the known 7-colorings. It would be interesting to obtain an 8-coloring of $G_{[1,b]}$ even for $b > 1.4$.

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