# Convexity-Increasing Morphs of Planar Graphs

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**Abstract**

We study the problem of *convexifying* drawings of planar graphs. Given any planar straight-line drawing of a 3-connected graph $G$, we show how to morph the drawing to one with convex faces while maintaining planarity at all times. Furthermore, the morph is *convexity increasing*, meaning that angles of inner faces never change from convex to reflex. We give a polynomial time algorithm that constructs such a morph as a composition of a linear number of steps where each step either moves vertices along horizontal lines or moves vertices along vertical lines.

## 1 Introduction

A *morph* between two planar straight-line drawings $\Gamma_0$ and $\Gamma_1$ of a graph $G$ is a continuous movement of the vertices from one to the other, with the edges following along as straight-line segments between their endpoints. A morph is *planar* if it preserves planarity of the drawing at all times. Motivated by applications in animation and in reconstruction of 3D shapes from 2D slices, the study of morphing has focused on finding a morph between two given planar drawings. The existence of planar morphs was established long ago [4, 15], followed by algorithms that produce good visual results [7, 8], and algorithms that find "piece-wise linear" morphs with a linear number of steps [1, 2].

Our focus is somewhat different, and more aligned with graph drawing goals—our input is a planar graph drawing and our aim is to morph it to a better drawing, in particular to a convex drawing. A morph *convexifies* a given straight-line graph drawing if the end result is a *convex graph drawing*, i.e. a planar straight-line graph drawing in which every face is a convex polygon.

We first observe that it is easy, using known results, to find a planar morph that convexifies a given graph drawing—we can just create a convex drawing with the same faces (assuming such a drawing exists), and morph to that specific drawing using the known planar morphing algorithms. However, ideally, convex angles should remain convex throughout a morph. We therefore impose the stronger condition that the morph be *convexity-increasing*, meaning that an angle of an inner face never switches from convex to reflex. Besides the theoretical goal of studying continuous motion that is monotonic in some measure (e.g. edge lengths [10]), another motivation comes from visualization—a morph of a graph drawing should maintain the user's "mental model" [13] which means changing as little as possible, and making observable progress towards a goal. Previous morphing algorithms fail to provide convexity-increasing morphs even if the target is a convex drawing because they all start by triangulating the drawing. This means that an original convex angle may be subdivided by new triangulation edges, so there is no constraint that keeps it convex. (An exception is Angelini et al. [2] which morphs a convex drawing to a convex drawing, preserving convexity.)

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This is an extended abstract of a presentation given at EuroCG'18. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

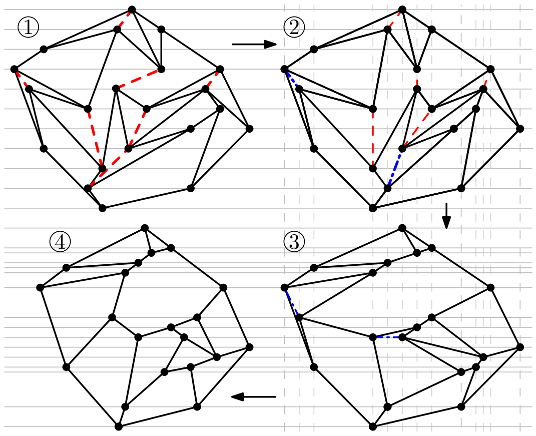


**Figure 1** A sequence of convexity-increasing morphs (horizontal, vertical, horizontal) that morph a straight-line drawing of a graph $G$ (drawn in black) into a strictly convex drawing of $G$.

**Our Results.** We give the first algorithm to convexify any straight-line planar drawing of a 3-connected graph via a planar convexity-increasing morph. We show a surprising stronger property—that the morph can be composed of a linear number of *horizontal* and *vertical* morphs. A *horizontal* morph moves all vertices at constant speeds along horizontal lines, and a *vertical* morph is defined similarly. See Figure 1. Orthogonality is a very desirable and well-studied criterion for graph drawing [6], in part because there is evidence that the human visual cortex comprehends orthogonal lines more easily [3, 14]. Similarly, it seems natural that orthogonal motion should be easier to comprehend, though morphing algorithms have so far not explored this criterion. To be precise, we prove the following theorem.

▶ **Theorem 1.1.** *Let $\Gamma$ be a planar straight-line drawing of a 3-connected graph $G$ on $n$ vertices. Then $\Gamma$ can be morphed to a strictly convex drawing via a sequence of convexity-increasing planar morphs each of which is either a horizontal morph or a vertical morph. If $\Gamma$ has a convex outer face then the number of morphs in the sequence is at most $r + 1$, where $r$ is the number of internal reflex angles in $\Gamma$. In general, the number of morphs in the sequence is at most $1.5n$. Furthermore there is an $O(n^{1+\omega/2})$ time algorithm to find the sequence of morphs, where $\omega$ is the matrix multiplication exponent.*

Due to space constraints, some proofs in this paper are only sketched or omitted entirely. Full proofs of all claims can be found in the full preprint [11].

## 2 Preliminaries

Two planar drawings of a graph $G$ have the *same combinatorial embedding* if they have the same clockwise cyclic ordering of edges around the outer face and around each inner face. We use the terms *convex*, *strictly convex*, *reflex* with their standard meanings. We say that a face of a planar graph drawing is *$y$-monotone* if the boundary of the face consists of two $y$-monotone chains. A chain is *$y$-monotone* if the $y$-coordinates of points along the chain are strictly increasing.

Horizontal and vertical morphs are special cases of *unidirectional morphs* [1] which move vertices along parallel lines, with each vertex moving at constant speed (different vertices are allowed to move at different speeds, and some may remain stationary). A unidirectional morph is completely specified by the initial and final drawings. We use the notation $\langle\Gamma_1, \Gamma_2\rangle$

to denote the linear morph from drawing $\Gamma_1$ to $\Gamma_2$. We use the following nice properties of unidirectional morphs; details are in the long version.

▶ **Lemma 2.1.** *[1, Lemma 13] If $\Gamma$ and $\Gamma'$ are two planar straight-line drawings of the same graph such that every line parallel to the $x$-axis crosses the same ordered sequence of edges and vertices in both drawings, then the linear morph from $\Gamma$ to $\Gamma'$ is planar.*

▶ **Lemma 2.2.** *Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three planar straight-line drawings where the linear morphs $\langle\Gamma_i, \Gamma_{i+1}\rangle$, $i = 1, 2$ are horizontal and planar. Then the linear morph $\langle\Gamma_1, \Gamma_3\rangle$ is a horizontal planar morph.*

▶ **Lemma 2.3.** *During a horizontal morph, an angle cannot change more than once between reflex and convex or vice versa. If $\langle\Gamma_1, \Gamma_2\rangle$ is a horizontal morph and any convex internal angle of $\Gamma_1$ is also convex in $\Gamma_2$ then the morph is convexity-increasing.*

▶ **Observation 2.4.** *[1, Lemma 13] If $\Gamma$ is a (not necessarily straight-line) planar graph drawing of a graph $G$ with all faces (including the outer face) $y$-monotone and $\Gamma'$ is another planar drawing of $G$ that has the same combinatorial embedding, the same $y$-coordinates of vertices, and has $y$-monotone edges, then every line parallel to the $x$-axis crosses the same ordered sequence of edges and vertices in both drawings.*

## 2.1 Redrawing with Convex Faces while Preserving $y$-Coordinates

We build upon an algorithm due to Hong and Nagamochi [9] that redraws a planar graph to have convex faces while preserving the $y$-coordinates of the vertices. Angelini et al. [2] strengthened the result to strictly convex faces. We limit ourselves to 3-connected graphs and improve the running time.

▶ **Lemma 2.5** (based on [9, 2])**.** *Let $\Gamma$ be a planar drawing of a 3-connected graph $G$ such that every face is $y$-monotone (including the outer face). Let $C$ be a strictly convex straight-line drawing of the outer face of $G$ such that every vertex of $C$ has the same $y$-coordinate as in $\Gamma$. Then there is a straight-line strictly convex drawing $\Gamma'$ of $G$ that has $C$ as the outer face and such that every vertex of $\Gamma'$ has the same $y$-coordinate as in $\Gamma$. Furthermore, $\Gamma'$ can be found in time $O(n^{\omega/2})$, where $\omega$ is the matrix multiplication exponent.*

We prove Lemma 2.5 in the long version using Tutte's graph drawing algorithm. This is quite different from the previous approaches, and gives the improved run-time.

Hong and Nagamochi [9] proved a version of Lemma 2.5, but did not guarantee a strictly convex drawing and gave a run-time of $O(n^2)$. Angelini et al. [2] strengthened the result to strictly convex faces by perturbing vertices to avoid angles of 180°. They did not analyze run-time. Our run time is $O(n^{1.5})$ without fast matrix multiplication, $O(n^{1.1865})$ with. Both [9] and [2] expressed their results in terms of *level planar drawings* of *hierarchical plane st-graphs*, and handled more generally the class of graphs that have [strictly] convex drawings.

# 3 Computing Convexity-Increasing Morphs

## 3.1 Morphing Drawings with a Convex Outer Face

To give some intuition about the proof, we first consider an easy case where the outer face of $\Gamma$ is strictly convex and all faces are $y$-monotone. Then we can immediately apply Lemma 2.5 with the outer face fixed to obtain a new straight-line strictly convex drawing $\Gamma'$ with all vertices at the same $y$-coordinates. The properties of unidirectional morphs can then be used to show that the morph from $\Gamma$ to $\Gamma'$ is planar, horizontal, and convexity-increasing.

A face $f$ is $y$-monotone if and only if it has only one *local maximum* and only one *local minimum*, where a vertex $v$ is a *local minimum* (*local maximum*) of face $f$ if the neighbors of $v$ in $f$ lie above $v$ (below $v$, respectively). A *local extremum* refers to a local minimum or a local maximum. We can augment a graph drawing to have $y$-monotone faces by adding $y$-monotone edges (not necessarily straight-line). This is a standard operation in upward planar (or "monotone") drawing [5, Lemma 4.1] [12, Lemma 3.1], but we need the stronger property that new edges are only incident to local extrema:

▶ **Proposition 3.1.** *Any straight-line planar graph drawing can be augmented to have $y$-monotone inner faces by adding edges such that each edge can be drawn as a $y$-monotone curve joining two local extrema in some face. Furthermore, these edges can be found in time $O(n \log n)$.*

This proposition allows us to prove the following:

▶ **Lemma 3.2.** *Let $\Gamma$ be a straight-line planar drawing with a convex outer face and no horizontal edge. There exists a horizontal planar morph to a straight-line drawing $\Gamma'$ such that $\Gamma'$ has a strictly convex outer face and every internal angle that is not a local extremum is strictly convex in $\Gamma'$. Furthermore, the morph is convexity-increasing, and can be found in time $O(n^{\omega/2})$, where $\omega$ is the matrix multiplication exponent.*

**Proof sketch.** We use Proposition 3.1 to augment $\Gamma$ with a set of edges $A$ such that $\Gamma \cup A$ is a planar drawing in which all faces are $y$-monotone, and any edge of $A$ goes between two local extrema in some inner face. This takes time $O(n \log n)$. Next, we apply Lemma 2.5 to obtain a new straight-line strictly convex drawing $\Gamma' \cup A'$ with all vertices at the same $y$-coordinates as in $\Gamma$. Finally, we apply the unidirectional properties to prove that the morph from $\Gamma$ to $\Gamma'$ is planar, horizontal, and convexity-increasing. Any internal angle of $\Gamma$ that is not a local extremum has no edge of $A$ incident to it, and thus is strictly convex in $\Gamma'$. The time required is $O(n^{\omega/2})$. ◀

Lemma 3.2 convexifies any *h-reflex* angle, where a reflex angle of inner face $f$ is *h-reflex* if it occurs at a vertex that has one neighbor in $f$ above and the other below (i.e., it is not a local extremum of $f$). To convexify the remaining reflex angles, the plan is to conceptually "turn the paper" by 90° and perform a vertical morph to make any *v-reflex* angle convex, where a reflex angle of inner face $f$ is called *v-reflex* angle if it occurs at a vertex that has one neighbor in $f$ to the left and the other to the right. The final aspect of the proof is to ensure that at each step there is at least one h-reflex or v-reflex angle, so that the algorithm makes progress in each step. To do this we strengthen Lemma 3.2:

▶ **Lemma 3.3.** *Let $\Gamma$ be a straight-line planar drawing with a convex outer face and no horizontal edge. There exists a horizontal planar morph to a straight-line drawing $\Gamma''$ such that*
*(i) the outer face of $\Gamma''$ is strictly convex,*
*(ii) every internal angle that is not a local extremum is convex in $\Gamma''$,*
*(iii) $\Gamma''$ has no vertical edge, and*
*(iv) if $\Gamma''$ is not convex, then it has at least one v-reflex angle.*
*Furthermore, the morph is convexity-increasing, and can be found in time $O(n^{\omega/2})$.*

**Proof sketch.** See Figure 2. We first apply Lemma 3.2 to obtain a morph from $\Gamma$ to a drawing $\Gamma'$ that satisfies (i) and (ii). If $\Gamma'$ satisfies all the requirements, we are done. Otherwise

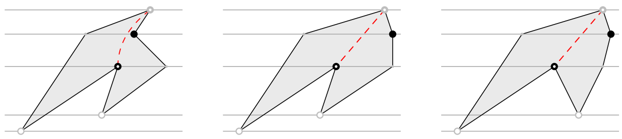

**Figure 2** (left) A face that is not $y$-monotone. (middle) The face after application of Lemma 3.2. There is a vertical edge and the single reflex vertex is not $v$-reflex. (right) After applying a horizontal shear transformation, the reflex vertex is $v$-reflex and there are no vertical edges.

we will achieve the remaining properties by choosing one reflex angle (necessarily a local extremum), say at vertex $u$ in inner face $f$, and applying a horizontal shear transformation to create a drawing $\Gamma''$ in which the angle at $u$ becomes v-reflex. Since shearing is an affine transformation, the convexity status of all angles is preserved. We then use the properties of unidirectional morphs to show that the morph from $\Gamma$ to $\Gamma''$ is planar, horizontal, and convexity-increasing. The morph can be found in time $O(n^{\omega/2})$. ◀

**Proof sketch of Theorem 1.1 for a convex outer face.** Apply Lemma 3.3 alternately in the horizontal and vertical directions until the drawing is convex. In each step there is at least one h-reflex or v-reflex angle that becomes convex. Thus the number of horizontal morphs is at most $r+1$ and the run-time is $O(n^{1+\omega/2})$. ◀

## 3.2 Morphing Drawings with a Non-convex Outer Face

In this section we outline the proof of Theorem 1.1 when the outer face is not convex. We augment the outer face of $\Gamma$ with new edges $A$ from its convex hull to obtain a drawing with a convex outer face. We apply the results from Section 3.1 to morph to a strictly convex drawing and then remove the edges of $A$ one-by-one. After each edge is removed we morph to a strictly convex drawing of the reduced graph using at most three horizontal or vertical morphs. Each edge $e \in A$ is part of the boundary of an inner face $f_e$ of $\Gamma \cup A$. We call $f_e$ the *pocket* of $e$. In order to remove edge $e$ we "pop" its pocket outward using the following lemma:

▶ **Lemma 3.4.** *Let $\Gamma$ be a strictly convex drawing of graph $G$, with an edge $e$ on the outer face. Suppose that $G - e$ is 3-connected. Then $\Gamma - e$ can be morphed to a strictly convex drawing of $G - e$ via at most three convexity-increasing morphs, each of which is horizontal or vertical. Furthermore, the morphs can be found in time $O(n^{\omega/2})$.*

See Figure 3 for the proof idea. Observe that each application of Lemma 3.4 increases the number of vertices of $G$ on the convex hull. Thus, by induction we obtain the proof of Theorem 1.1. We have used at most 3 horizontal and vertical morphs per pocket. To obtain the initial strictly convex drawing of $\Gamma \cup A$ we require at most $n$ morphs. Thus, the total number of morphs is bounded by $4n$. In the full version we decrease the total number of morphs to $1.5n$. The run time of the algorithm is $O(n^{1+\omega/2})$.

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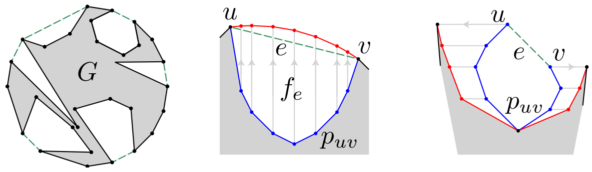


**Figure 3** (left) Schematic of the convex drawing of $G \cup A$. Graph $G$ is depicted in gray, edges of $A$ are dashed, and the pockets are white. (middle),(right) Illustration of Lemma 3.4. If a pocket $p_{uv}$ is $x$-monotone we can pop it out (middle), otherwise we first make it $x$-monotone (right).

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