QPTAS and Subexponential Algorithm for Maximum Clique on Disk Graphs *

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Abstract

A disk graph is the intersection graph of closed disks in the plane. We show the structural result that a disjoint union of cycles is the complement of a disk graph if and only if at most one of those cycles is of odd length. From that, we derive the first QPTAS and subexponential algorithm running in time $2^{\tilde{O}(n^{2/3})}$ for Maximum Clique on disk graphs. In contrast, the problem is unlikely to have such algorithms on intersection graphs of filled ellipses or filled triangles.

1 Introduction

Intersection graphs for many different families of geometric objects have been widely studied due to their practical applications and rich structural properties [13]. Among the most studied ones are (unit) *disk graphs*, which are intersection graphs of closed (unit) disks in the plane with applications ranging from sensor networks to map labeling.

Clark et al. [8] gave a polynomial-time algorithm for Maximum Clique on unit disk graphs. The complexity of the problem on general disk graphs is unfortunately still unknown. Ambühl and Wagner [1] gave a simple 2-approximation algorithm for Maximum Clique on general disk graphs, showed the problem to be APX-hard on intersection graphs of ellipses and gave a $9\rho^2$-approximation algorithm for filled ellipses of aspect ratio at most ρ.

Results. We show that the disjoint union of two odd cycles is not the complement of a disk graph, providing an infinite family of forbidden induced subgraphs, an analogue to the work of Atminas and Zamaraev on unit disk graphs [2]. Using this property we give a QPTAS and a subexponential-time algorithm for Maximum Clique on disk graphs. Finally, we show that for filled ellipses or filled triangles, there is a constant $\alpha > 1$ for which an α-approximation algorithm running in subexponential time is highly unlikely.

Definitions and notations. For two integers $i \leqslant j$, let $[i, j]$ be the set $\{i, i+1, \ldots, j-1, j\}$ and $[i]$ the set $[1, i]$. For a subset S of vertices of a graph, let $N(S)$ be the open neighborhood of S and $N[S]$ the set $N(S) \cup S$. The *2-subdivision* of a graph G is the graph H obtained by subdividing each edge of G twice. The *co-2-subdivision* of G is the complement of H. The *co-degree* of G is the maximum degree of its complement. A *co-disk* is the complement of a disk graph. For two distinct points x and y in the plane, we denote by $\ell(x, y)$ the unique line going through x and y, and by $\text{seg}(x, y)$ the closed straight-line segment whose endpoints are x and y. For a segment s with positive length, let $\ell(s)$ be the unique line containing s.

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2 Disk graphs with co-degree 2

We fully characterize the degree-2 complements of disk graphs by showing the following.

▶ **Theorem 1.** *A disjoint union of cycles is the complement of a disk graph if and only if the number of odd cycles is at most one.*

We only show the first part of this theorem, i.e., the union of two disjoint odd cycles is not the complement of a disk graph. As disk graphs are closed under taking induced subgraphs, two vertex-disjoint odd cycles in the complement of a disk graph have to be linked by at least one edge. The second part, i.e., how to represent the complement of the disjoint union of even cycles and one odd cycle, is deferred to the full version of this abstract [4].

2.1 The disjoint union of two odd cycles is not co-disk

A *proper representation* is a disk representation where every edge is witnessed by a proper intersection of the two corresponding disks, i.e., the interiors of the two disks intersect. It is easy to transform a disk representation into a proper one (of the same graph) where, additionally, no three disk centers are aligned. With this assumption, we can show that in a representation of a $K_{2,2}$ with the four centers in convex position, both non-edges have to be *diagonal*.

▶ **Lemma 2.** *In a disk representation of $K_{2,2}$ with the four centers in convex position, the non-edges are between vertices corresponding to opposite centers in the quadrangle.*

A useful consequence of the previous lemma is the following.

▶ **Corollary 3.** *In any disk representation of $K_{2,2}$ with centers c_1, c_2, c_3, c_4 with the two non-edges between the vertices corresponding to c_1 and c_2, and between c_3 and c_4, it should be that $\ell(c_1, c_2)$ intersects seg(c_3, c_4) or $\ell(c_3, c_4)$ intersects seg(c_1, c_2).*

We can now prove the main result of this section thanks to the previous corollary, parity arguments, and some elementary properties of closed plane curves.

▶ **Theorem 4.** *The complement of the disjoint union of two odd cycles is not a disk graph.*

Proof. Let s and t be two positive integers and $G = \overline{C_{2s+1} + C_{2t+1}}$ the complement of the disjoint union of two cycles of lengths $2s+1$ and $2t+1$. Assume that G is a disk graph. Let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be the cycle embedded in the plane formed by $2s+1$ (resp. $2t+1$) straight-line segments joining the consecutive centers of disks along the first (resp. second) cycle. We number the segments of $\mathcal{C}_1$ from S_1 to S_{2s+1}, and the segments of $\mathcal{C}_2$, from S'_1 to S'_{2t+1}.

For the i-th segment S_i of $\mathcal{C}_1$, let a_i be the number of segments of $\mathcal{C}_2$ intersected by the line $\ell(S_i)$ prolonging S_i, let b_i be the number of segments S'_j of $\mathcal{C}_2$ such that $\ell(S'_j)$ intersects S_i, and let c_i be the number of segments of $\mathcal{C}_2$ intersecting S_i. For the second cycle, we define similarly a'_j, b'_j, c'_j. The quantity $a_i + b_i - c_i$ counts the number of segments of $\mathcal{C}_2$ which can possibly represent a $K_{2,2}$ with S_i according to Corollary 3. Since G is a disk graph, $a_i + b_i - c_i = 2t + 1$ for every $i \in [2s + 1]$. Otherwise there would be at least one segment S'_j of $\mathcal{C}_2$ such that $\ell(S_i)$ does not intersect S'_j *and* $\ell(S'_j)$ does not intersect S_i.

Observe that a_i is an even integer since $\mathcal{C}_2$ is a closed curve. Also, $\Sigma_{i=1}^{2s+1} a_i + b_i - c_i = (2t + 1)(2s + 1)$ is an odd number, as the product of two odd numbers. This implies that $\Sigma_{i=1}^{2s+1} b_i - c_i$ shall be odd. $\Sigma_{i=1}^{2s+1} c_i$ counts the number of intersections of the two closed curves $\mathcal{C}_1$ and $\mathcal{C}_2$, and is therefore even. Hence, $\Sigma_{i=1}^{2s+1} b_i$ shall be odd. Observe that $\Sigma_{i=1}^{2s+1} b_i = \Sigma_{j=1}^{2t+1} a'_j$ by reordering and reinterpreting the sum from the point of view of the segments of $\mathcal{C}_2$. Since the a'_j are all even, $\Sigma_{i=1}^{2s+1} b_i$ is also even; a contradiction. ◀

3 Algorithmic consequences

As a clique in a graph G is an independent set in $\overline{G}$, we focus on Maximum Independent Set on graphs without two vertex-disjoint odd cycles with no edges connecting them.

3.1 QPTAS

The odd cycle packing number $\mathrm{ocp}(H)$ of a graph H is the maximum number of vertex-disjoint odd cycles in H. The condition that $\overline{G}$ does not contain two vertex-disjoint odd cycles with no edges between them is not the same as saying that $\mathrm{ocp}(\overline{G}) = 1$. Otherwise, we could directly use the PTAS on graphs H with n vertices and $\mathrm{ocp}(H) = o(n/\log n)$ by Bock et al. [3], which does not need the odd cycle packing as an input. This is important, since finding a maximum odd cycle packing is NP-hard [11]. Using Theorem 4, we can show the following lemma, which spares us having to determine the odd cycle packing number.

▶ **Lemma 5.** *Let H be a graph with n vertices, whose complement is a disk graph. If $\mathrm{ocp}(H) > n/\log^2 n$, then H has a vertex of degree at least $n/\log^4 n$.*

If $\overline{G}$ has no vertex of degree at least $n/\log^4 n$, by Lemma 5, we know that $\mathrm{ocp}(\overline{G}) \leqslant n/\log^2 n = o(n/\log n)$ and run the PTAS of Bock et al. If $\overline{G}$ has a vertex v of degree at least $n/\log^4 n$ (it may still hold that $\mathrm{ocp}(\overline{G}) = o(n/\log n)$), we branch on v: either we include v in our solution and remove $N[v]$, or we discard v. The complexity of this is given by $F(n) \leqslant F(n-1) + F(n - n/\log^4 n)$, which solves to the running time below.

▶ **Theorem 6.** *For any $\varepsilon > 0$, Maximum Clique can be $(1+\varepsilon)$-approximated in time $2^{O(\log^5 n)}$, when the input is a disk graph with n vertices.*

3.2 Subexponential algorithm

An *odd cycle cover* is a subset of vertices whose deletion makes the graph bipartite. Györi et al. [9] showed that graphs with small odd girth have small odd cycle cover. This can be seen as relativizing the fact that odd cycles do not have the Erdős-Pósa property. Bock et al. [3] turned the non-constructive proof into a polynomial-time algorithm.

▶ **Theorem 7** ([9] and [3])**.** *Let H be a n-vertex graph with no odd cycle shorter than δn. H has a polynomial-time computable odd cycle cover of size at most $(48/\delta)\ln(5/\delta)$.*

We start by showing three variants of an algorithm.

▶ **Theorem 8.** *Let G be a disk graph with n vertices and Δ, c the maximum degree and odd girth of $\overline{G}$. Maximum Clique has a branching or can be solved, up to a polynomial factor, in time: (i) $2^{\tilde{O}(n/\Delta)}$ (branching), (ii) $2^{\tilde{O}(n/c)}$ (solved), (iii) $2^{O(c\Delta)}$ (solved).*

Proof. We look for a maximum independent set in $\overline{G}$. For (i), let v be a vertex of degree Δ in $\overline{G}$. We branch on v: either we include v in our solution and remove $N[v]$, or discard v. The complexity is given by $F(n) \leqslant F(n-1) + F(n-(\Delta+1))$, which solves to (i). This does not give a $2^{\tilde{O}(n/\Delta)}$-time algorithm as the maximum degree might drop. We do the branching as long as it is *good enough* and finish with the algorithms corresponding to (ii) and (iii).

For (ii) and (iii), let C be the cycle of length c, it can be found in polynomial time. By Theorem 7, with $\delta = c/n$, we find an odd cycle cover X in $\overline{G}$ of size $\tilde{O}(n/c)$ in polynomial time. We exhaustively guess in time $2^{\tilde{O}(n/c)}$ the intersection I of an optimum solution with X and finish by finding, in polynomial time, a maximum independent set in the bipartite graph $\overline{G} - (X \cup N(I))$. The total complexity of this case is $2^{\tilde{O}(n/c)}$, which shows (ii).

For (iii), $\overline{G} - N[C]$ is bipartite as $\overline{G}$ contains no two vertex-disjoint odd cycles with no edges between them. As every vertex in $\overline{G}$ has degree at most Δ, it holds that $|N[C]| \leqslant c(\Delta - 1) \leqslant c\Delta$. Indeed, a vertex of C can only have $c(\Delta - 2)$ neighbors outside C. We guess the intersection of the optimal solution with $N[C]$ and find the maximum independent set in a bipartite graph (a subgraph of $\overline{G} - N[C]$), which can be done in total time $2^{O(c\Delta)}$. ◀

The structure of G affects the bounds in Theorem 8 as follows.

▶ **Corollary 9.** *Let G be a disk graph with n vertices.* Maximum Clique *can be solved in time: (a) $2^{\tilde{O}(n^{2/3})}$, (b) $2^{\tilde{O}(\sqrt{n})}$ if the maximum degree of $\overline{G}$ is constant, (c) polynomial, if both the maximum degree and the odd girth of $\overline{G}$ are constant.*

On general graphs, (b) is the hardest case for Maximum Clique. Moreover, the win-win strategy of Corollary 9 can be directly applied to solve Maximum Weighted Clique.

4 Intersection graphs of filled ellipses and filled triangles

For Maximum Clique on intersection graphs of (*non-filled*) ellipses and triangles, APX-hardness was shown by Ambühl and Wagner [1]. Their reduction also implies that there is no subexponential algorithm for this problem, unless the ETH fails [10]. They claim that their hardness result extends to filled ellipses since *"intersection graphs of ellipses without interior are also intersection graphs of filled ellipses"*. Unfortunately, this claim is incorrect.

▶ **Theorem 10.** *There is a graph which has an intersection representation with ellipses without their interior, but has no intersection representation with convex sets.*

Figure 1 shows a counterexample and the argument is similar to the one used by Brimkov et al. [5], which was in turn inspired by the construction by Kratochvíl and Matoušek [12].

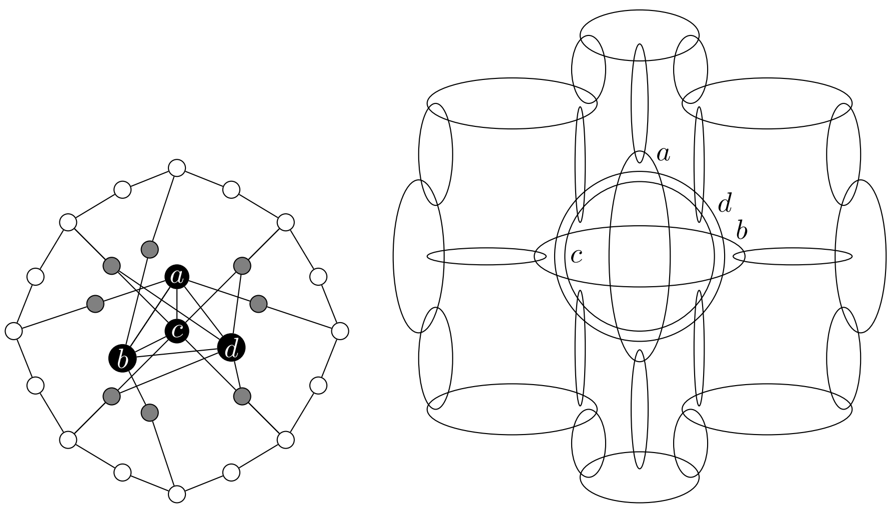

Figure 1 A graph (left), which has a representation with empty ellipses (right) but no representation with convex sets.

Fortunately, we can show that the hardness result *does* hold for filled ellipses (and filled triangles) with a different reduction. Our construction can be seen as streamlining the ideas of Ambühl and Wagner [1] and, we believe, is simpler.

▶ **Theorem 11.** *There is a constant $\alpha > 1$, such that for every $\varepsilon > 0$,* Maximum Clique *on intersection graphs of filled ellipses or filled triangles has no α-approximation algorithm running in subexponential time $2^{n^{1-\varepsilon}}$, unless the ETH fails. For ellipses, this holds even when they have arbitrarily small eccentricity and arbitrarily close value of major axis.*

This contrasts with the subexponential algorithm and QPTAS for eccentricity 0 (disks). It also subsumes [6] (where NP-hardness is shown for connected shapes contained in a disk of radius 1 and containing a concentric disk of radius $1 - \varepsilon$ for arbitrarily small $\varepsilon > 0$).

We first show the lower bound for Maximum Weighted Independent Set on the class of all 2-subdivisions, and, hence, the same for Maximum Weighted Clique on all co-2-subdivisions. Then we show that intersection graphs of filled ellipses or of filled triangles contain all co-2-subdivisions.

The following inapproximability result for Maximum Independent Set on bounded-degree graphs was shown by Chlebík and Chlebíková [7]. As their reduction is almost linear, the PCP of Moshkovitz and Raz [14] boosts this hardness result from ruling out polynomial-time up to ruling out subexponential time $2^{n^{1-\varepsilon}}$ for any $\varepsilon > 0$.

▶ **Theorem 12** ([7, 14])**.** *There is a constant $\beta > 0$ such that* Maximum Independent Set *on graphs with n vertices and maximum degree Δ cannot be $1+\beta$-approximated in time $2^{n^{1-\varepsilon}}$ for any $\varepsilon > 0$, unless the ETH fails.*

▶ **Theorem 13.** *There is a constant $\alpha > 1$ such that for any $\varepsilon > 0$,* Maximum Independent Set *on the class of all the 2-subdivisions has no α-approximation algorithm running in subexponential time $2^{n^{1-\varepsilon}}$, unless the ETH fails.*

Proof. Let G be a graph with maximum degree Δ, n vertices $v_1, \ldots, v_n$ and m edges $e_1, \ldots, e_m$. Let H be its 2-subdivision. The $2m$ vertices in $V(H) \setminus V(G)$, representing edges, are called *edge vertices* and denoted by $v^+(e_1), v^-(e_1), \ldots, v^+(e_m), v^-(e_m)$, as opposed to the other *original vertices* of H. If $e_k = v_iv_j$ is an edge of G, then $v^+(e_k)$ (resp. $v^-(e_k)$) has two neighbors: $v^-(e_k)$ and v_i (resp. $v^+(e_k)$ and v_j).

There is a maximum independent set S which contains exactly one of $v^+(e_k), v^-(e_k)$ for every $k \in [m]$. S cannot contain both $v^+(e_k)$ and $v^-(e_k)$ as they are adjacent. If S contains neither $v^+(e_k)$ nor $v^-(e_k)$, then adding $v^+(e_k)$ to S and potentially removing the other neighbor of $v^+(e_k)$, can only increase the size of the independent set. Hence S contains m edge vertices and $s \leqslant n$ original vertices, and there is no larger independent set in H.

Assume an approximation with ratio $\alpha := 1 + \frac{2\beta}{(\Delta+1)^2}$ for Maximum Independent Set on 2-subdivisions running in subexponential time, where ratio $1+\beta > 1$ is not attainable for Maximum Independent Set on graphs of maximum degree Δ (Theorem 12). On instance H, this algorithm would output a solution with m' edge vertices and s' original vertices. This solution can be easily (in polynomial time) transformed into an at-least-as-good solution with m edge vertices and s'' original vertices forming an independent set in G. We assume that $s'' \geqslant n/(\Delta + 1)$ since for any independent set of G, we can obtain an independent set of H consisting of the same set of original vertices and m edge vertices. Since $m \leqslant n\Delta/2$ and $s'' \geqslant n/(\Delta+1)$, we obtain $m \leqslant s''\Delta(\Delta+1)/2$ and $2m/(\Delta+1)^2 \leqslant s''\Delta/(\Delta+1)$. From $\frac{m+s}{m+s''} \leqslant \alpha$ and $\Delta \geqslant 3$, we easily get that $s \leqslant s''(1 + \beta)$, contradicting the Theorem 12. ◀

Finally, we can prove the following lemma; see Figure 2 for an example with filled triangles. This, together with Theorem 13, proves Theorem 11.

▶ **Lemma 14.** *The class of intersection graphs of filled triangles or filled ellipses contains all co-2-subdivisions.*

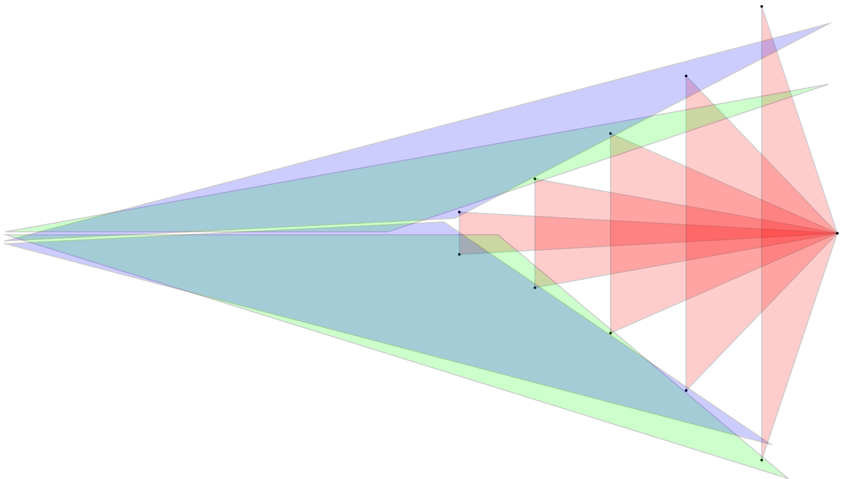

Figure 2 A co-2-subdivision of a graph with 5 vertices (in red) represented with triangles. Two edges are represented: between vertices 1 and 4 (in green) and between vertices 2 and 3 (in blue).

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