$L(2,1)$-labeling of disk intersection graphs

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Abstract

In this paper we study the problem of $L(2,1)$-labeling of intersection graphs of disks. An $L(2,1)$-labeling is a mapping from the vertex set of the graph to non-negative integers, in which labels assigned to adjacent vertices differ by at least 2, and labels assigned to vertices at distance 2 are different. The span of an $L(2,1)$-labeling is the difference between the maximum and the minimum label used, and the span $\lambda(G)$ of a graph G is the minimum span of an $L(2,1)$-labeling of G. We show that if G is an intersection graph of disks, then $\lambda(G) \leq \frac{4}{5}\Delta(G)^2 + 25\Delta(G) + 22$, where $\Delta(G)$ denotes the maximum degree in graph G.

1 Introduction

Problems arising in frequency assignment in radio networks gave raise to many interesting graph-theoretical questions, especially concerning various variants of graph coloring. Notable and well-studied members of a big family of such problems are channel assignment problem [14], T-coloring [19], distance-constrained labeling [9, 14, 19], and $L(p,q)$-labeling [8, 2]. They are interesting for their potential applications [23], and purely theoretical properties.

In this paper, we consider one of such problems, i.e. the $L(2,1)$-labeling problem. It asks for a vertex labeling with non-negative integers, in which adjacent vertices get labels that differ by at least two, and vertices at distance two get different labels. The chromatic parameter related to this problem is the $L(2,1)$-*span* of a graph G, denoted by $\lambda(G)$, which is the minimum possible difference between the largest and the smallest label used by an $L(2,1)$-labeling of G. Griggs and Yeh [8] showed that for every G it holds that $\lambda(G) \leq \Delta(G)^2 + 2\Delta(G)$, where $\Delta(G)$ denotes the maximum vertex degree in G. Moreover, they conjectured that $\lambda(G) \leq \Delta(G)^2$ for every graph G with $\Delta(G) \geq 2$. This conjecture attracted a considerable attention and upper bounds were successfully improved, e.g. Gonçalves [6] showed an algorithm finding and $L(2,1)$-labeling of any graph with $\Delta(G) \geq 3$, whose span is at most $\Delta(G)^2 + \Delta(G) - 2$. Using a non-constructive method, Havet *et al.* [10] settled the "delta-square conjecture" in affirmative for graphs with $\Delta(G) \geq 10^{69}$. For graphs with smaller maximum degree the problem remains open. Another open direction is finding a constructive proof of the conjecture.

What makes the delta-square conjecture even more interesting is the fact that we know only two graphs G, which satisfy the equality $\lambda(G) = \Delta(G)^2$: they are the Petersen graph and the Singleton-Hoffman graph. Recently Lu [15] presented an infinite family of graphs G with $\lambda(G) = \Delta(G)^2 - \Delta(G) + 1$, which is the largest value for any known infinite family.

Besides the results for general graphs, also restricted graph classes received a considerable attention. For example it is known that $\lambda(G) \leq \Delta(G)+2$ if G is a tree [8], $\lambda(G) \leq 2\Delta(G)+23$ if G is planar [22], and $\lambda(G) \leq \frac{d-2}{d-1}\Delta(G)^2+2\Delta(G)$, if G is $K_{1,d}$-free [20]. We refer the reader to the survey by Calamoneri [2] for more information about $L(2,1)$-labeling and related problems.

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In this paper we focus on the class of intersection graphs of disks in the Euclidean plane called disk graphs or DG in short. This class with its subclass where all disks are required to have equal diameter (shortly denoted as UDG) are among the most extensively studied classes of geometric intersection graphs, both from the combinatorial and the algorithmic point of view [3, 5, 7, 18]. They are also especially interesting and natural in the context of $L(2,1)$-labeling, since they are the simplest class used for modeling radio networks [16, 17, 23]. Since unit disk graphs are $K_{1,6}$-free (see Fig.1), the bound by Shao *et al.* [20] implies that $\lambda(G) \leq \frac{4}{5}\Delta(G)^2 + 2\Delta(G)$. Fiala *et al.* [4] considered offline and online algorithms for $L(2,1)$-labeling of disk and unit disk graphs. Among other results, they have shown that $\lambda(G) \leq 18\omega(G)$ for $G \in UDG$, where $\omega(G)$ denotes the cardinality of the largest clique in G. This clearly implies that $\lambda(G) \leq 18\Delta(G) + 18$ for such graphs.

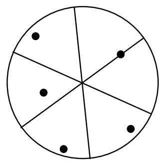

Figure 1 A maximal independent set in the neighborhood of a vertex of UDG. (Points represent the centers of the disks. Disks with centers in each region form a clique. For any point we can rotate the partition so that it is on the boundary of two regions, and then we can only add 4 other points to the independent set.)

In this paper we continue the line of research started by Fiala *et al.* [4], Shao *et al.* [20] and Junosza-Szaniawski *et al.* [12]. We solve the delta-square conjecture for disk graphs with maximum degree at least 126, by proving the following theorem.

▶ **Theorem 1.1.** *For any disk graph G with maximum degree $\Delta(G)$ we have $\lambda(G) \leq \frac{4}{5}\Delta(G)^2 + 25\Delta(G) + 22$.*

To the best of our knowledge, this is the first non-trivial upper bound for $\lambda(G)$ if G is a disk intersection graph, without any further assumption on the radii of the disk in a geometric representation.

Throughout the paper, we assume that the input disk intersection graph is given along with its geometric representation. Note that this assumption is important, as the problems of recognizing unit disk graphs [1] and disk graphs [11] are NP-hard. Actually, the problem of recognizing unit disk graphs is known to be $\exists\mathbb{R}$-complete [13], which is a strong evidence that it may not even be in NP.

2 Preliminaries

For a graph $G = (V, E)$, by $\Delta(G)$ and $\omega(G)$ we denote, respectively, the maximum degree and the size of the maximum clique in G. By $\bar{G}$ we denote the *complement* of G, i.e. a graph with the vertex set V and the edge set $\binom{V}{2} \setminus E$. For vertices u, v of G, by $\mathrm{d}_G(u,v)$ we denote the number of edges on the shortest u-v–path in G (i.e., the distance between u and v in the graph G). By $N(v)$ we denote the *neighborhood* of the vertex v, i.e. the set of vertices u with $\mathrm{d}_G(u,v) = 1$. By $N^2(v)$ we denote the set of vertices u with $\mathrm{d}_G(u,v) = 2$.

A function $c\colon V \to \mathbb{N}_0$ is called an $L(2,1)$-*labeling* of $G = (V, E)$, if

1. $|c(v) - c(w)| \geq 1$ for all $v, w \in V$ such that $\mathrm{d}_G(u,w) = 2$,
2. $|c(v) - c(w)| \geq 2$ for all $v, w \in V$ such that $\mathrm{d}_G(v,w) = 1$.

A *span* of an $L(2,1)$-labeling c of G is the difference between the maximum and the minimum label used by c (note that some labels may not be used at all). An $L(2,1)$-*span* of G, denoted by $\lambda(G)$, is the minimum possible span in an $L(2,1)$-labeling of G. Note that the number of labels that might be used in an $L(2,1)$-labeling with the minimum span is $\lambda(G) + 1$.

For $u, v \in \mathbb{R}^2$, by $\mathrm{dist}(v,u)$ we denote the euclidean distance between u and v. For $r \in \mathbb{R}$ and $v \in \mathbb{R}^2$ by $D(v,r)$ we denote the set $\{p \in \mathbb{R}^2 : \mathrm{dist}(v,p) \leq r\}$, i.e., the disk with a center in v and the radius r. For a set $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ of disks in the plane, we define their *intersection graph* $G = (V, E)$, whose vertex set is $\{v_1, v_2, \ldots, v_n\}$, and the vertices $v_i v_j$ are adjacent if and only if $D_i \cap D_j \neq \emptyset$. We will often identify the vertices of G with the centers of the disks in $\mathcal{D}$. Notice that $v_i v_j \in E$ if and only if $\mathrm{dist}(v_i, v_j)$ is at most the sum of the radii of D_i and D_j.

For such a graph G, the set $\mathcal{D}$ is called *geometric representation by intersection disks*, or a *representation* in short. A graph G is called a *disk intersection graph*, or a *disk graph* in short, if it has a geometric representation by intersecting disks. If a graph G admits a geometric representation, where all disks have the same radius, we say that G is a *unit disk intersection graph*, or *unit disk graph*. The classes of disks intersection graphs and unit disks intersection graphs are denoted by DG and UDG, respectively.

We will also use the celebrated Turán Theorem.

▶ **Theorem 2.1** (Turán [21]). *For integers $d \geq p$, every K_p-free graph with d vertices has at most $\frac{p-2}{p-1}\frac{d^2}{2}$ edges.*

3 General disk intersection graphs

In this section we prove Theorem 1.1. Consider a disk graph $G = (V, E)$ along with its geometric representation by the collection of intersecting disks $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$. Let v_i be the center and r_i be the radius of D_i. We will identify points $v_1, v_2, \ldots, v_n$ with the corresponding vertices of G. Moreover, we assume that disks are ordered by non-increasing radius, i.e. $r_1 \geq r_2 \geq \ldots \geq r_n$.

Consider any vertex $v_i \in V$. We start with defining two special types of vertices in $N^2(v_i)$. We say that v_j is an LL-neighbor of v_i, or, equivalently, $v_j \in N^2_{LL}(v_i)$, if $v_j \in N^2(v_i)$, $r_j \geq r_i$ and there exists a disk D_k intersecting both D_i and D_j such that $r_k \geq r_i$ ("LL" stands for Large-Large, as both D_j and D_k are larger than D_i). Analogously, we say that v_j is an SL-neighbor of v_i, or $v_j \in N^2_{SL}(v_i)$, if $v_j \in N^2(v_i)$, v_j is not an LL-neighbor of v_i, $r_j \geq r_i$, and there exists a disk D_k intersecting both D_i and D_j such that $r_k < r_i$ (here "SL" stands for Small-Large, as D_k is smaller than D_i and D_j is larger than D_i).

Now we want to bound the cardinalities of $N^2_{SL}(v_i)$ (in Lemma 3.1) and $N^2_{LL}(v_i)$ (in Lemma 3.2) of each vertex v_i of G.

▶ **Lemma 3.1.** *$|N^2_{SL}(v_i)| \leq 22\omega(G)$, for any vertex v_i in a disk graph G.*

Proof. Let $v_j \in N^2_{SL}(v_i)$. Since D_i and D_j do not intersect, we have $dist(v_i, v_j) > r_i + r_j \geq 2r_i$. On the other hand there exists a disk D_k intersecting both D_i and D_j, such that $r_k \leq r_i$. Thus we obtain

$$\begin{aligned} dist(v_i, v_j) \leq& dist(v_i, v_k) + dist(v_k, v_j) \leq (r_i + r_k) + (r_k + r_j) \\ \leq& r_i + 2r_k + r_j \leq 3r_i + r_j. \end{aligned} \tag{$\star$}$$

We partition $\mathbb{R}^2 - D(v_i, 2r_i)$ into 22 regions, which will form cliques, in the following manner. First we divide $\mathbb{R}^2 - D(v_i, 2r_i)$ by a circle with the center in v_i and radius $2t \cdot r_i$, where $t = \frac{1}{\sqrt{2-\sqrt{2}}} \approx 1.3$. Then we partition the ring $D(v_i, 2tr_i) - D(v_i, 2r_i)$ into 8 congruent bounded regions, and $\mathbb{R}^2 - D(v_i, 2tr_i)$ into 14 congruent unbounded regions, as presented on Figure 2. In the remainder of the proof, we will show that disks with the radius at least r_i, whose centers lie inside one region, form a clique. Clearly this will imply the lemma. It

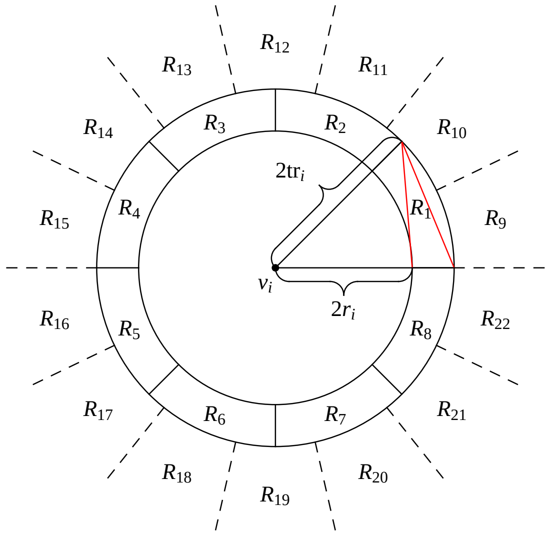

Figure 2 Partition of $\mathbb{R}^2 - D(v_i, 2r_i)$ into 22 regions: $R_1, R_2, \ldots, R_{22}$, $(t = \frac{1}{\sqrt{2-\sqrt{2}}})$.

is straightforward to verify that the diameter of each bounded region is at most

$$\begin{aligned}\text{diam} &= \max\left\{2r_i\sqrt{\left(t\cos\frac{\pi}{4}-1\right)^2+\left(t\sin\frac{\pi}{4}\right)^2}, 2t_i\sqrt{\left(t\cos\frac{\pi}{4}-t\right)^2+\left(t\sin\frac{\pi}{4}\right)^2}\right\}\\ &= \max\left\{2r_i\sqrt{t^2-t\sqrt{2}+1}, 2r_i\sqrt{(2-\sqrt{2})t^2}\right\}.\end{aligned}$$

Since $t = \frac{1}{\sqrt{2-\sqrt{2}}}$, we obtain $diam = \max\left\{2r_i\sqrt{\frac{1}{2-\sqrt{2}} - \frac{\sqrt{2}}{\sqrt{2-\sqrt{2}}} + 1}, 2r_i\right\} = 2r_i$. Thus any two disks with centers in one bounded region and radii at least r_i must intersect.

Now consider two disks D_p, D_q, whose centers lie in an unbounded region and $v_p, v_q \in N^2_{SL}(v_i)$. We consider three cases. First, suppose that both v_p and v_q are at distance at most $4r_i$ from v_i. Then the distance between v_p and v_q is at most

$$\max\left\{2r_i\sqrt{(2\cos\frac{\pi}{7}-t)^2+(2\sin\frac{\pi}{7})^2}, 2r_i\sqrt{(2\cos\frac{\pi}{7}-2)^2+(2\sin\frac{\pi}{7})^2}\right\} < 2r_i.$$

Since $r_p, r_q \geq r_i$, we know that $r_p + r_q \geq 2r_i$, and thus D_p and D_q intersect.

Now consider the second case where v_p is at distance at most $4r_i$ from v_i, but the distance between v_q and v_i is greater than $4r_i$ (see Figure 3). Let Q be the intersection point of the line containing points v_i and v_q, and the circle with radius $4r_i$, centered at v_i. Let $D_Q := D(Q, r_q - dist(Q, v_q))$ be the disk with the center Q, contained in and tangent to D_q. Notice that the radius of D_Q is at least r_i (since, by ($\star$), we obtain $r_q - \text{dist}(Q, v_q) \geq \text{dist}(v_i, v_q) - 3r_i - \text{dist}(Q, v_q) = \text{dist}(v_i, Q) - 3r_i = 4r_i - 3r_i = r_i$). Hence, from the previous case, we know that D_Q and D_p intersect. Since $D_Q \subseteq D_q$, disks D_q and D_p also intersect.

For the last case, suppose that v_p and v_q both at distance greater than $4r_i$ from v_i. As in the previous case we define the disk $D_Q \subseteq D_q$ and analogously $D_P \subseteq D_p$. The radii of both D_Q and D_P are at least r_i, so the disks intersect. Consequently D_q and D_p intersect. ◀

Now we consider the cardinality of $N^2_{LL}(v_i)$.

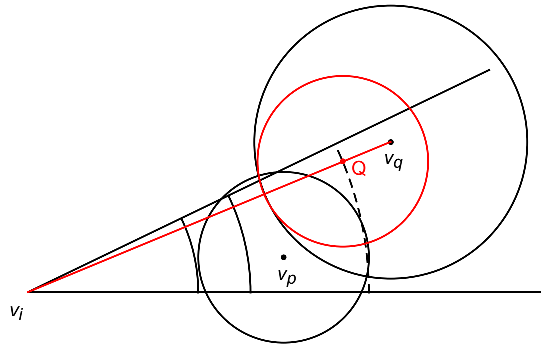

Figure 3 Case 2. Location of the point Q.

▶ **Lemma 3.2.** $|N^2_{LL}(v_i)| \leq \frac{4}{5}\Delta(G)^2$, *for any vertex* v_i *in a disk graph* G.

Proof. Let v_i be a vertex of graph G. Let H be a graph induced by the neighbors of v_i corresponding to disks with radius at least r_i. We define $\Delta := \Delta(G)$ and $d := |V(H)| \leq \Delta$. Notice that v_i cannot have 6 independent neighbors - recall the argument showing that UDG are $K_{1,6}$-free. Thus the graph H is $\overline{K_6}$-free, and hence $\overline{H}$ is K_6-free. By Theorem 2.1, the maximum number of edges in $\overline{H}$ is $\frac{4}{5}\frac{d^2}{2}$. Therefore the number of edges in H is at least $\binom{d}{2} - \frac{4}{5}\frac{d^2}{2} = \frac{d^2}{10} - \frac{d}{2}$.

The obvious upper bound on the number of all possible vertices in $N^2_{LL}(v_i)$ is $d(\Delta - 1)$. Each edge in H reduces this number by two. Thus we obtain the following upper bound on $|N^2_{LL}(v_i)|$:

$$d(\Delta - 1) - 2\left(\frac{d^2}{10} - \frac{d}{2}\right) = d\Delta - \frac{d^2}{5}.$$

One can easily verify this expression is maximized for $d = \Delta$. Hence $|N^2_{LL}(v_i)| \leq \frac{4}{5}\Delta^2$. ◀

Now Theorem 1.1 is an easy consequence of Lemmas 3.1 and 3.2.

Proof of Theorem 1.1. Consider a greedy algorithm labeling vertices of G, ordered by non-increasing radii of disks in the geometric representation. Let v_i be a vertex of G and let $V' = \{v_1, v_2, \ldots, v_{i-1}\}$ be the set of vertices that are already labeled. We will compute the maximum possible number of labels that cannot be used to label a vertex v_i. Each neighbor of v_i that belongs to V' blocks at most 3 labels, which gives at most $3\Delta(G)$ labels in total. Each vertex in $V' \cap N^2(v_i)$ blocks 1 label. Recall that we can partition the set $V' \cap N^2(v_i)$ into two subsets – $N^2_{SL}(v_i)$ and $N^2_{LL}(v_i)$. By Lemma 3.1, $N^2_{SL}(v_i)$ blocks at most $22\omega(G) \leq 22(\Delta(G) + 1)$ labels in total. By Lemma 3.2, $N^2_{LL}(v_i)$ blocks at most $\frac{4}{5}\Delta(G)^2$ labels in total. Hence the number of labels that cannot be used for v_i is at most $\frac{4}{5}\Delta(G)^2 + 25\Delta(G) + 22$. ◀

4 Conclusion

A very natural question to ask is whether the upper bounds presented in this paper are tight. It is interesting to look for non-trivial constructions of families of (unit) disk graphs with large $L(2,1)$-span, compared to their maximum degree. It is even more interesting, since very little is known about the topic. To the best of our knowledge the largest lower bound is equal to $2\Delta(G)$ and is obtained by a $2k$-th power of a cycle of length $4k+1$, so by the unit disk graph. Clearly this bound is very far from the known upper bounds. It is also very interesting if we can actually force a quadratic span in general disk intersection graphs.

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