A Fully Polynomial Time Approximation Scheme for the Smallest Diameter of Imprecise Points

Vahideh Keikha[1], Maarten Löffler[2], and Ali Mohades[3]

1 **Department of Mathematics and Computer Science, University of Sistan and Baluchestan, Zahrdan, Iran,** `va.keikha@aut.ac.ir.com`

2 **Department of Information and Computing Sciences, Utrecht University, Utrecht, The Netherlands,** `m.loffler@uu.nl`

3 **Department of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran,** `mohades@aut.ac.ir`

Abstract

Given a set $D = \{d_1, ..., d_n\}$ of imprecise points modeled as disks, the minimum diameter problem is to locate a set $P = \{p_1, ..., p_n\}$ of fixed points, where $p_i \in d_i$, such that the furthest distance between any pair of points in P is as small as possible. This introduces a tight lower bound on the size of the diameter of any instance P. In this paper, we present a fully polynomial time approximation scheme (FPTAS) for this problem that runs in $O(n^3\epsilon^{-2})$ time, where the input is a set of disjoint disks.

1 Introduction

One of the most extensively quantitative techniques used to deal with uncertainty of the input is the *Region-based model*, where the input is a set $R = \{r_1, r_2, \ldots, r_n\}$ of regions and each region represents an imprecise point. In this model, the minimum diameter problem was studied by Löffler and van Kreveld [1], who presented a PTAS in $O(n^{3\pi/\sqrt{\epsilon}})$ time, where the uncertainty of the input was modeled by arbitrary disks. In the same paper, the authors also presented an exact algorithm that runs in $O(n \log n)$ time and computes an upper bound on the diameter (maximum diameter problem). Recently, a new approximation algorithm was presented for this problem, where the uncertainty of the input is modeled by convex objects in d-dimensional space [5]. The presented algorithm runs in $O(2^{\epsilon^{-d}}\epsilon^{-2d}n^3)$ time. The minimum diameter problem is also studied in other models of uncertainty [2–4, 6, 9].

Contribution. We formulate our problem as follows. We are given a set $D = \{d_1, d_2, \ldots, d_n\}$ of imprecise points modeled as disjoint disks; choose a set $P = \{p_1, ..., p_n\}$ of points, where $p_i \in d_i$, such that the size of the diameter of P is as small as possible among all choices for each p_i.

As for the result, we present an FPTAS for this problem which runs in $O(n^3\epsilon^{-2})$ time (Section 3).

2 Preliminaries

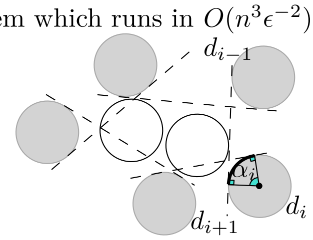


Figure 1 Critical sequence (gray disks).

In terms of definitions and terminology, we will follow Löffler and van Kreveld [1]. For a given set $D = \{d_1, d_2, \ldots, d_n\}$ of imprecise points modeled as disks, an *extreme disk* $d_i \in D$ has a line ℓ tangent to some point on the boundary of d_i, where no other disk can have its interior completely on the same side of d_i, unless it is tangent to ℓ, as illustrated in Figure 1. The *critical sequence* Δ_D is the set of all extreme disks of D. Without loss of generality, suppose the elements of Δ_D are ordered clock-wise. The ordered set Δ_D can be found in $O(n \log n)$ time [8].

34th European Workshop on Computational Geometry, Berlin, Germany, March 21–23, 2018.
This is an extended abstract of a presentation given at EuroCG'18. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

In the minimum diameter problem it is possible that the diameter occurs at several pairs at the same time, and many points are involved in the diameter, such that moving any of them would increase the distance between at least one pair of them. This property makes the problem difficult (see Figure 3(right)). In this situation, the diameter constructs a graph, the *star graph*. A star graph G is defined as $G = (P^*, E)$, where $P^* = \{b_1, ..., b_m\}$ $(m \leq n)$ is a collection of points, such that $b_i \in d_i$ for some i, and all the elements of E have equal length, which is the optimal diameter and denoted by d^*. Since this is the diameter, no two points can be more than $|d^*|$ away from each other. It follows that all the elements of E must intersect each other, and the path makes an angle of at most 60° at each vertex. Thus the degree of each vertex of G can be at most two, and each element of P^* (with degree 2) is in *balance* between its neighbors, that is, it could move closer to one neighbor but only by moving further from the other neighbor. Note that we can remove any vertex in this graph that can come closer to some element without moving further from another element. Also notice that all n points could be involved in such a construction, where none of the points can be moved without increasing the diameter. The elements of P^* are called *bends*. Indeed, computing the exact positions of the bends is a difficult problem. Any two adjacent disks on Δ_D introduce a common tangent line (see Figure 1). The *extreme arc* α_i of an extreme disk d_i is defined by two disks d_{i-1} and d_{i+1} that are neighbors of d_i on Δ_D, such that the endpoints of α_i are the touching points of the common tangent lines, as illustrated in Figure 1. Note that α_i is the part of the boundary of disk d_i that must contain b_i. Let r_i and c_i denote the radius and the center of d_i, respectively. The distance from a point x on disk d_i to a disk d_j is the minimum distance between x and any point on d_j. The distance between two arcs α_i and α_j (resp. two disks d_i and d_j) is the minimum distance from any point on α_i (resp. d_i) to any point on α_j (resp. d_j).

▶ **Observation 2.1.** *For a given set D of n unit disks with $n \geq 3$, the smallest diameter $|d^*|$ is at least* 0.28 *and for $n \geq 7$, $|d^*| \geq 2$.*

The observation can be easily proved by considering Figure 2.

▶ **Lemma 2.2.** *Let $D = \{d_1, ..., d_n\}$ be a set of disjoint disks, and let $|\alpha_i|$ denote the size of the constructed angle where the two lines through the endpoints of the extreme arc α_i of extreme disk d_i meet the center. Then $\sum_{i=1}^{n} |\alpha_i| = 2\pi$.*

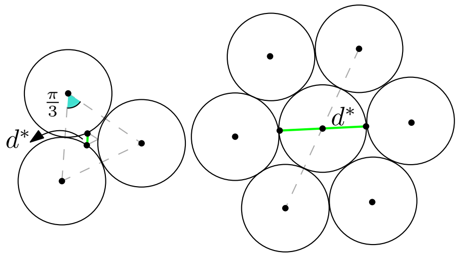


Figure 2 (left) For $n \geq 3$, $|d^*| \geq 0.28$. (right) For $n \geq 7$, $|d^*| \geq 2$.

Cherry disks. For any disk d_i that shares bend b_i on the star graph, there always exist two other disks $d_j, d_k \in \Delta_D$ with $j \neq k$, such that bend b_i is in balance between them, that is b_i could move closer to one of them but only by moving further from the other one. We call d_j and d_k the *cherry disks* of d_i. An example is illustrated in Figure 3. If a disk $d_i \in \Delta_D$ does not have two cherry disks, d_i cannot introduce a bend on the star graph. It is an interesting open question to compute all the possible cherry disks efficiently, since then the minimum diameter problem can be formulated as a second order cone program, and its optimal solution can be computed in quadratic time [7]. This results in an $O(n^2 n!)$ time exact algorithm.[1]

For any disk $d_i \in \Delta_D$, let e_{ij} and e_{ik} denote the intersection of disk d_i by line segments $c_i c_j$ and $c_i c_k$, respectively (see Figure 3(right)), such that d_j and d_k are the cherry disks of

[1] In a model of computation that we can exactly compute the roots of any constant degree polynomials.

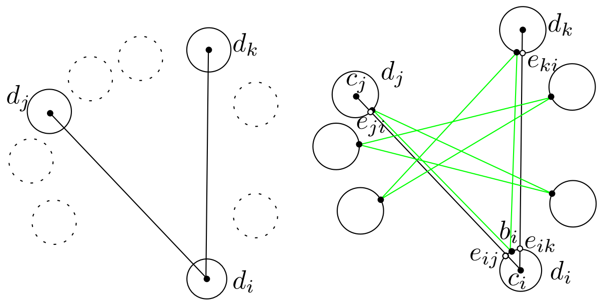


Figure 3 (left) Extreme disk $d_i \in D$ has two cherry disks d_j and d_k. (right) All the cherry disks of D; the constructed star graph on them is shown in green. Bend b_i is located in the interval $[e_{ij}, e_{ik}]$.

d_i. The points e_{ij} and e_{ik} are the startpoint and the endpoint of the arc on which bend b_i on α_i is in balance between the cherry disks of d_i (which we will prove later). We denote this interval by $[e_{ij}, e_{ik}]$. Obviously d_i is also a cherry disk for both d_j and d_k.

▶ **Lemma 2.3.** *Let d_j and d_k with $j < k$ denote the cherry disks of d_i. Then b_i is located in the interval $[e_{ij}, e_{ik}]$.*

Proof. Suppose this is false. Then bend b_i is strictly located either before e_{ij} or after e_{ik} (on the boundary of d_i). Consider the case where b_i is strictly located before the position e_{ij} (the other case is similar). In this case, at least b_j is strictly located after e_{ji} (on the cw ordering of the boundary of d_j). But then the two disks (or even one) which determine the position of b_j on d_j must be located between d_i and d_j. Let d_p and d_q denote these disks. Since b_j is strictly located after e_{ji}, at least one of d_p and d_q is different from d_i. Clearly, $b_i b_k$, $b_j b_p$ and $b_j b_q$ are some edges of the star graph. But $b_j b_p$ and $b_j b_q$ never intersect the edge $b_i b_k$. This gives a contradiction with the fact that all the edges of the star graph are pairwise intersecting. ◀

3 Minimum diameter problem

If the disks are not unit but still disjoint, Observation 2.1 holds if the smallest disk is a unit disk.

3.1 Unit disks

From Lemma 2.2 we know that if D consists of disjoint disks, the total sum of the angles of extreme arcs equals 2π. First, suppose D consists of disjoint unit disks. We proceed by *covering* the boundary of a unit disk U by all the extreme arcs of set Δ_D, such that they just intersect at the endpoints, as illustrated in Figure 4(a). This covering is indeed a translation transformation. We decompose the boundary of U into smaller, equal-length *sub-arcs* by regularly inserting $2\pi/\epsilon$ points. Then, for any disk d_i, the added points on the boundary of U which is covered by α_i will be transferred to the boundary of d_i, as illustrated in Figure 4(a). Consequently the extreme arcs get divided into sub-arcs of length at most ϵ.

Recall that P^* denote the optimal point set. Let $P' = \{p_1, ..., p_m\}$ denote the optimal point set restricted to the endpoints of the sub-arcs. The set P' minimizes the furthest distance between any pair of points on P' among all possible choices for P' [2]. Let d denote the diameter of P'. As said before, d^* denote the optimal diameter of P^*. We will show that d approximates d^* within a factor $(1+\epsilon)$.

For any disk $d_i \in \Delta_D$, we define the *optimal sub-arc* α_i^* that includes (if any) the bend b_i. Also, α_i^* minimizes the difference of distances of the endpoints of α_i^* to the approximated cherry disks of

[2] We use the name P for the set of all the candidate points of set D, where $P' \subseteq P$. It is easy to observe that the diameter d' of P' equals the diameter d of P.

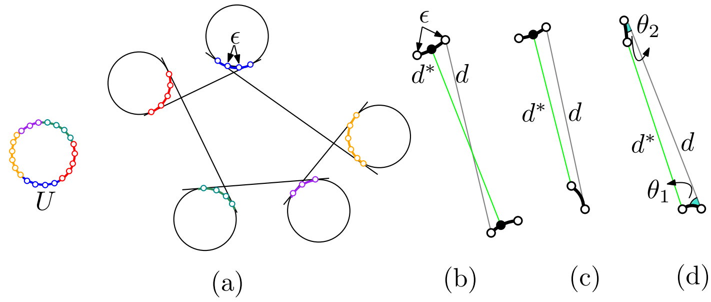


Figure 4 (a) Subdivisions of the extreme arcs into sub-arcs of length ϵ. (b-d) The configuration of the approximated diameter (gray) and the optimal diameter (green).

d_i, where the diameter which realizes by this selection of α_i^*, is as small as possible. Note that in the optimal solution, the length of each α_i^* equals 0.

We will postpone the discussion of computing the optimal sub-arcs, and we first consider how set P' approximates the minimum diameter. Note that the optimal diameter d^* at least equals the largest distance between any two optimal sub-arcs (where the distance between two arcs α_i and α_j is the minimum distance from any point on α_i to any point on α_j). Thus, we show that for any two optimal sub-arcs which include the vertices of the potential minimum diameter, the ratio of the smallest distance to the furthest distance equals $(1+\epsilon)$. There exist two configurations to consider the ratio of the smallest distance to the furthest distance of a pair of optimal sub-arcs.

- The case where d and d^* intersect each other (see Figure 4(b)). Let d_1 and d_2 (resp. d_1^* and d_2^*) denote the two segments which are determined on d (resp. d^*) by intersection with d^* (resp. d), such that d_1 and d_1^* form a triangle, where the endpoints of its base are located on an optimal sub-arc.

  Since the length of the optimal sub-arc is at most equal to ϵ, by the triangle inequality we have $|d_1^*|+\epsilon > |d_1|$ and $|d_2^*|+\epsilon > |d_2|$, and since $|d^*| \geq 2$, $|d| \leq |d^*|(1+\epsilon)$.
- The case where d and d^* do not intersect each other, in which case d^* selects its two vertices at the endpoints of its optimal sub-arcs, or d^* selects one vertex at the middle of one of its optimal sub-arcs (as illustrated in Figure 4(c,d)). Let θ_1 and θ_2 denote the angles between d and (tangents of) the optimal sub-arcs, then $|d| \leq \epsilon(\cos\theta_1 + \cos\theta_2) + |d^*|$. This again gives us $|d| \leq |d^*|(1+\epsilon)$.

Computing the optimal sub-arcs. Let m denote the number of extreme disks of D. We show that for any disk $d_i \in \Delta_D$, we can find α_i^* efficiently. For any disk $d_i \in \Delta_D$ we first select point p_i which is chosen to be one of the endpoints of the sub-arcs of α_i. This is the initialization of set P'. Then, during the algorithm, we try to move each element of P' to its best position, so that the final set P' minimizes the diameter among all possible choices for P'. Indeed, for any disk d_i we look for the optimal sub-arc α_i^*, where one of the endpoints of α_i^* determines one element of P'.

In each step of the algorithm we start by computing the diameter of P'. Let d' denote the diameter of P' with p_i and p_j as the vertices. If p_i (or p_j) is not yet in balance, we move it forward among the endpoints of the sub-arcs of α_i in the direction that the size of d' is decreasing. In each possible movement, we update the size of d', and stop moving p_i, when in the next movement, the distance of p_i to any other point p_k will be greater than the current size of d'. Let $d'' < d'$ denote the diameter with a vertex at p_j. Then we move p_j forward in the direction that the size of d'' is decreasing, and also we update the size of d'' in each movement, until in the next movement, the distance of p_j to any other point p_l is greater than the current size of d''. We also repeat this procedure for p_k and p_l, respectively, by computing the corresponding diameter with a vertex at p_k and p_l, respectively. We stop this step when we have checked/corrected the position of all the elements of P', each of which one time.

In the second step, we again start by computing the diameter of P'. We continue above procedure, until we check the position of all the elements of P'. Since the vertices of the diameter may already be in balance, it is not always possible to move them to reduce the diameter. In the following we prove that it is always possible to reduce the value of the diameter after $O(m\epsilon^{-1})$ consecutive steps of the algorithm.

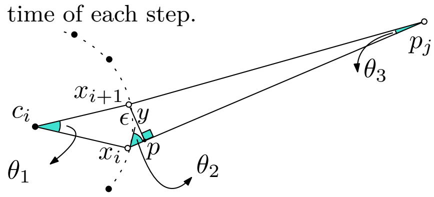
In the last step of the algorithm, we only can check the position of all the elements of P', while no other movement is possible. This way we have approximated the cherry disks of any disk d_i, and also one endpoint of the optimal sub-arc α_i^*. The other endpoint is the one which is closer to both cherry disks of d_i.

▶ **Lemma 3.1.** *After at most $O(m\epsilon^{-1})$ steps of the algorithm, the size of d' reduces by a factor of $\sqrt{2}/2$.*

Proof. In the worst case, in each step, we only could move one point p_a to its balanced position. Then in the next step, at least another point p_b, with $b \neq a$ can be moved (otherwise the algorithm will be terminated) which could not be moved in the previous step. This point can only be the point which previously was in balance between p_a and another point p_c. Then we may go back to move p_b, and then p_a and p_c, if they make a bend with its two cherry disks. Since the distances of the bend from its cherry disks are only decreasing, in at most $O(\epsilon^{-1})$ consecutive steps of the algorithm, either the algorithm stops, or we can move a new point which is distinct from p_a, p_b and p_c. Consequently, after at most $O(m\epsilon^{1-})$ steps, we have changed the position of all the elements of P', and since we have reduced all the furthest distances of the bends from the corresponding cherry disks, the size of the diameter is decreased. Now we clarify the changes on the size of the diameter that occur during the algorithm. Let p_i and p_j denote the vertices of the diameter d' which we have reduced its value. Thus we at least move one vertex of the diameter from a position x_i to x_{i+1}.

Let θ_1 denote the angle subtended by the arc with length ϵ at the center of d_i, and let θ_2 denote the determined angle by the intersection of p_jx_i and the tangent line of d_i at x_i, and let θ_3 denote the angle between p_jx_i and p_jx_{i+1}, as illustrated in the below Figure. Notice that the size of the angles θ_2 and θ_3 change during the algorithm. Also let y denote the height of triangle $x_ix_{i+1}p_j$ from the triangle's vertex x_{i+1} to the base p_jx_i, and let p denote the intersection point of y and p_jx_i. Since $|y| < |x_ix_{i+1}|$ and, $|x_ix_{i+1}| < 2$ and $|p_jx_i| = |d'| \geq 2$, the angle $|\theta_3| < 45°$. Consequently $\frac{|y|}{|d'|} < \frac{\sqrt{2}}{2}$. Also since $|p_jx_{i+1}| = |d''| \geq 2$, $|x_ix_{i+1}| < 2$ and $|\theta_3| < 45°$, the angle $|\theta_2| > 45°$. Then we have $\frac{|y|}{|d''|} < \frac{\sqrt{2}}{2}$ and $\frac{|x_ip|}{|x_ix_{i+1}|} < \frac{\sqrt{2}}{2}$, and thus $\frac{|x_ip|\cdot|y|}{|x_ix_{i+1}|\cdot|d''|} < \frac{2}{4}$. Since $\frac{|y|}{|x_ix_{i+1}|} < \frac{\sqrt{2}}{2}$, $|x_ip| < \frac{1}{\sqrt{2}}|d''|$ and since $|d''| < |d'|$ we have $|x_ip| < \frac{1}{\sqrt{2}}|d'|$ [3]. ◀

The importance of the reduced value from the diameter is on the convergence of the iterative process. Also $2 \leq |d^*| < |d_{max}|$, where d_{max} denote the maximum diameter of D (it can be computed in $O(n \log n)$ time [1]). Obviously the same bound also holds for d'. Consequently, the algorithm will be terminated after $O(m\epsilon^{-1}(\log_{\sqrt{2}}|d_{max}|)$ steps. Since $\log_{\sqrt{2}}|d_{max}|$ is a constant, we omit it from the total running time. Now we consider the running time of each step.



Since we know the cw order of the elements of P', the diameter of P' can be computed in linear time in each step of the algorithm. In each movement of any element p_i of P', we should be careful for not increasing the size of the diameter with a vertex at p_i. Thus we costs $O(m + m\epsilon^{-1})$ for each element in one step. The later m is the time costs to check whether the corresponding element gets in balance or not. Thus the algorithm takes $O(m^3\epsilon^{-1}(1+\epsilon^{-1}))$ time.

▶ **Lemma 3.2.** *For any disk $d_i \in \Delta_D$, computed α_i^* includes possible bend b_i.*

Proof. Suppose this is false. Then b_i is located on a sub-arc α_i' which is distinct from α_i^*. Then either we have passed over this sub-arc during the algorithm, or we did not find it and we stopped. Let d_j and d_k denote the approximated cherry disks of d_i by the algorithm.

In the first case, at both endpoints of α_i', the computed distance of d_i to both d_j and d_k must be greater than our current diameter, while we have found a solution with strictly a smaller size. This contradicts the optimality of the computed minimum diameter.

In the second case, since α_i' and α_i^* are distinct, at least one computed cherry disk for α_i' has to be distinct from d_j or d_k (if not; $\alpha_i' = \alpha_i^*$, and we are done). But then we could move p_i to

[3] Notice that this lemma holds for a set of arbitrary disks where the smallest disk is a unit disk, since we do not let the length of the sub-arcs (which is ϵ) on the smallest disk to be as large as π, $|y|$ must always be smaller than 2.

reduce the distance of d_i to at least one of d_j and d_k. This contradicts the stop criterion of the algorithm. ◀

3.2 Disks with different size

In the case where D consists of arbitrary disjoint disks, the total sum of the angles of the extreme arcs still equals 2π, but with the idea we used on unit disks, the optimal sub-arcs will not necessarily have the same length. In this case, we first apply the presented constant factor approximation algorithm [1] for the minimum diameter problem on a set of disks. The presented algorithm approximates the smallest diameter within a constant factor c in linear time. Let d' denote the approximated smallest diameter of D within factor c. We define $h = \epsilon \cdot c \cdot |d'|$ as the new length of the sub-arcs. Note that if h is greater than the extreme arc of a disk d_i, we consider α_i instead of a sub-arc of length h. In this case, the total sum of the lengths of the extreme arcs is bounded by $|2\pi(cd'/2)|$, since any circle whose radius is greater than $|cd'/2|$ will share an extreme arc with less curvature (and thus with less arc length), also any circle whose radius is less than $|cd'/2|$ will share an extreme arc with a shorter arc length (see Figure 5). Thus the maximum number of the points that approximates the extreme arcs is bounded by $\frac{2\pi(cd'/2)}{h}$, which is in $O(\epsilon^{-1})$. Since computed sub-arcs admit the same length h, the considered ratio of the furthest distance to the smallest distance between the optimal sub-arcs (in Section 3.1) still holds, and the presented algorithm works in $O(n^3\epsilon^{-2})$ time.

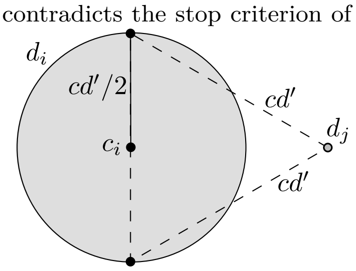


Figure 5 The maximum possible length for an extreme arc appears between two disks d_i and d_j with $|r_i| = |cd'|/2$ and $|r_j| \approx 0$. Thus the total sum of the extreme arcs is bounded by $|2\pi(cd'/2)|$. Note that the subtended angle of a sub-arc cannot equals π, it is supposed so to compute the upper bound.

▶ **Theorem 3.3.** *Given a set of n disjoint disks, the problem of choosing a point on the boundary of each disk such that the diameter of the resulting point set is as small as possible can be approximated within a factor $(1+\epsilon)$ in $O(n^3\epsilon^{-2})$ time.*

References

1 Largest bounding box, smallest diameter, and related problems on imprecise points. *Computational Geometry*, 43(4):419 – 433, 2010.

2 P. K. Agarwal, S. Har-Peled, and K. R. Varadarajan. Approximating extent measures of points. *Journal of the ACM (JACM)*, 51(4):606–635, 2004.

3 M. E. Consuegra and G. Narasimhan. Geometric avatar problems. In *LIPIcs-Leibniz International Proceedings in Informatics*, volume 24, 2013.

4 R. Fleischer and X. Xu. Computing minimum diameter color-spanning sets is hard. *Information Processing Letters*, 111(21):1054–1056, 2011.

5 M. Ghodsi, H. Homapour, and M. Seddighin. *Approximate Minimum Diameter*, pages 237–249. Springer International Publishing, Cham, 2017.

6 W. Ju, C. Fan, J. Luo, B. Zhu, and O. Daescu. On some geometric problems of color-spanning sets. *Journal of Combinatorial Optimization*, 26(2):266–283, 2013.

7 M. S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret. Applications of second-order cone programming. *Linear algebra and its applications*, 284(1-3):193–228, 1998.

8 T. Nagai, S. Yasutome, and N. Tokura. Convex hull problem with imprecise input and its solution. *Systems and Computers in Japan*, 30(3):31–42, 1999.

9 D. Zhang, Y. M. Chee, A. Mondal, A. K. Tung, and M. Kitsuregawa. Keyword search in spatial databases: Towards searching by document. In *Data Engineering*, pages 688–699, 2009.