# Protecting a highway from fire*†

**Rolf Klein[1], David Kübel[1], Elmar Langetepe[1], and Barbara Schwarzwald[1]**

1 **Department of Computer Science, University of Bonn**

## Abstract

Suppose a fire spreads at speed 1 along a highway given by a horizontal line in the $L_1$ plane. A fighter is tasked to protect the highway by building barriers along or perpendicular to the highway with a building speed $v$. The fighter can move without delay or additional costs between different construction sites. We show that $v > 1.5$ is necessary and $v > \frac{2+\sqrt{5}}{\sqrt{5}} = 1.8944\ldots$ is sufficient.

## 1 Introduction and problem statement

In fire fighting theory and practice, different tasks are important: quenching or enclosing a fire, and protecting objects from fire; see [6]. Results in continuous and discrete models differ significantly; see [1–5, 7].

In our scenario, a horizontal highway of infinite length in the $L_1$ plane must be protected from a fire originating from $(0,0)$ at speed 1 in all directions. The highway is considered as protected if no point on the highway is ever touched by the fire. For protection, a fire fighter may build barriers at speed $v$ while the fire is spreading. Barriers are impassable for the fire, but can only be built at locations where the fire has not arrived, yet.

One way to protect the highway would be to enclose the fire by barriers in some fashion. This model has been investigated e. g. by Bressan [2]. He considers a fighter that can build arbitrary curves and is able to fly between construction sites without delay and free of costs. For the $L_2$ plane, he showed that a building speed of $v > 1$ is necessary and $v > 2$ is sufficient. Despite serious effort, this gap is still open.

We discuss a different model and obtain better bounds, where $v > 1.5$ is necessary and $v > 1.8944\ldots$ suffices. In our model the flying fighter is tasked to protect the highway by building barriers along, or perpendicular to, the highway in the $L_1$ plane. Note that in this model the highway is protected if we can guarantee that the barriers along the highway can always be built before the fire arrives there. Thus, the purpose of the perpendicular barriers is only to slow down the fire. Namely, the fire has to move up and down to overcome the vertical barrier, while the fighter can set it up in one go; see e. g. barrier $d_2$ in Figure 1. This involves that $d_2$ must be ready where the fire crawls upwards. However, it need not be built a second time, when the fire crawls downwards along the backside. The question is, whether these savings allow to improve on the naive approach, which only builds horizontal barriers along the highway and obviously requires $v \geq 2$.

## 2 Model

In our model, the highway is the $x$-axis and the fire originates from the point $(0,0)$ and continuously expands over time equally in all directions with speed 1 according to the $L_1$ metric. Since we allow the fire to start at the highway, we allow an arbitrarily small head-start of barrier of length $s$ into both directions along the highway.

* This work has been supported by DFG grant Kl 655/19 as part of a DACH project.
† We thank all anonymous reviewers of EuroCG for their valuable comments.

34th European Workshop on Computational Geometry, Berlin, Germany, March 21–23, 2018.

A *barrier set* is described by two series of pairs $(a_i, b_i)$ and $(c_i, d_i)$, one for each direction. While $a_i$ and $c_i$ denote the distance to the previous barrier in the series (or to $(0,0)$ for $i = 1$), $b_i$ and $d_i$ denote the length of the vertical barrier; see Figure 1 for an example of a general barrier set. Note that this barrier set also includes horizontal barriers along the highway.

A necessary condition for a barrier set solution with speed $v$ can be obtained as follows. Assume that the final system of barriers has been built and then the fire spreads. At any moment of time $t$, the total amount of barrier *consumed* by the fire must not exceed $v \cdot t$. Here we consider a piece of the barriers as consumed, as soon as the fire has reached this piece for the first time; see Figure 1.

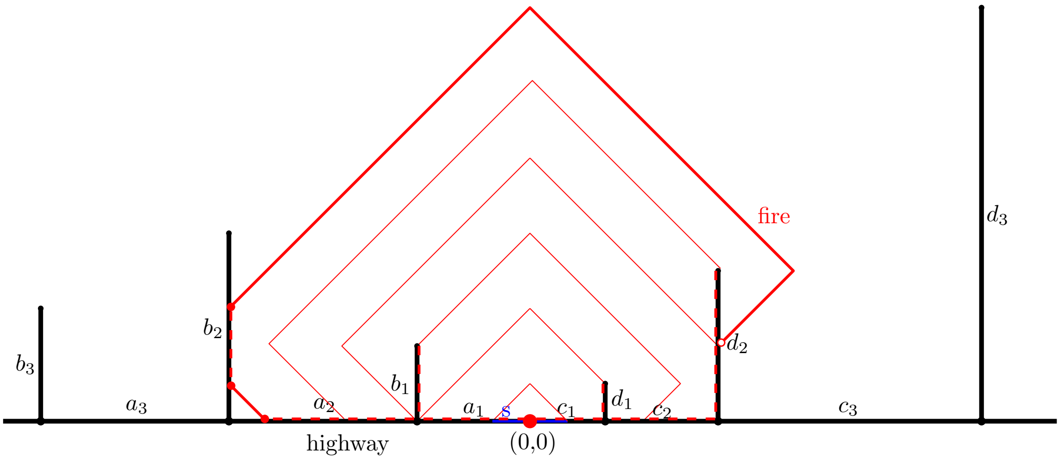


**Figure 1** A snapshot of the fire burning along a barrier set. Dashed lines mark the consumed pieces of the barriers. The fire burns along barriers at 4 places, namely twice at $b_2$ and once at $a_2$ and $d_2$. However, the current consumption is only 3, since the fire burning along the back of $d_2$, which had already been reached, does not count.

Thus, we define the *total consumption* and *consumption-ratio* for a barrier set $\mathcal{S}$:

$$\mathcal{C}_\mathcal{S}(t) \quad := \quad \text{length of all pieces of barriers} \in \mathcal{S} \text{ consumed by the fire until time t}$$
$$\mathcal{Q}_\mathcal{S}(t) \quad := \quad \frac{\mathcal{C}_\mathcal{S}(t)}{t}\,.$$

As the fire spreads over the barrier set, it represents a geodesic $L_1$ circle. At some points in time, the structure of its boundary changes. Between such events, the derivative of $\mathcal{C}_\mathcal{S}(t)$ is an integer given by the number of places, where the fire burns along a part of the barrier set for the first time. We call this value $k$ the *current consumption* and the time interval between the events a $k$*-interval*. In Figure 1, there are four boundary points, but only three of them count, thus the current consumption is 3.

Note that all these definitions can easily be restricted to either side of the barrier set, e. g. denoted by $\mathcal{Q}^l_\mathcal{S}(t)$ and $\mathcal{Q}^r_\mathcal{S}(t)$. Obviously, $\mathcal{Q}_\mathcal{S}(t) = \mathcal{Q}^l_\mathcal{S}(t) + \mathcal{Q}^r_\mathcal{S}(t)$.

A barrier set is then called *viable* for any speed $v$ such that $v > \max_t \mathcal{Q}_\mathcal{S}(t)$. Viability is necessary and sufficient, too, as Lemma 2.1 shows. Due to space limitations, we omit the proof of this technical lemma.

▶ **Lemma 2.1.** *For any barrier set that is viable for a speed $v$, there exists a building plan for this barrier set with speed $v + \delta$, where $\delta > 0$ can be arbitrarily small.*

The extra speed $\delta$ is required if we assume that the fighter cannot build single points and has to build pieces of positive length one at a time instead.

The question then obviously is: What is the minimum speed $v$ for which a viable barrier set exists?

## 3 A lower bound of $v > 1.5$

First, we show that not all possible barrier sets have to be considered.

▶ **Lemma 3.1** (Increasing barrier lengths). *For any barrier set $\mathcal{S}$ viable for a speed $v$, there exists a barrier set consisting of vertical barriers of strictly increasing length in each direction which is also viable for $v$. That is, $b_{i+1} > b_i$ and $d_{i+1} > d_i$, $\forall i \geq 1$.*

**Proof.** Assume $\mathcal{S}$ contains a vertical barrier $w_2$ that is at most as long as the previous one $w_1$ in that direction. Then consider the barrier set $\mathcal{S}' = \mathcal{S} \setminus w_2$. Removing $w_2$ does not decrease the shortest $L_1$ path from the fire origin $(0, 0)$ to any point in the plane including points on all other barriers, as those paths are prolonged by $w_1$. However, $w_2$ is no longer consumed and hence the total consumption $\mathcal{C}_{\mathcal{S}'}(t) \leq \mathcal{C}_{\mathcal{S}}(t)$ for all $t$ and therefore $\mathcal{Q}_{\mathcal{S}'}(t) \leq \mathcal{Q}_{\mathcal{S}}(t)$. ◀

▶ **Theorem 3.2** (Lower Bound). *There exists no viable barrier set for any speed $v \leq 1.5$.*

**Proof.** The idea of the proof is to show that the ratios $\mathcal{Q}^{l,r}_{\mathcal{S}}(t)$ are always larger than 0.5, for $t$ greater than some threshold and sometimes exceed 1 for arbitrarily large $t$. We start with the latter.

Assume there exists a barrier set $\mathcal{S}^*$ which is viable for $v < 1.5$. For every time $t_0 > 0$ there must exist some point in time $t_1 > t_0$, where the current consumption is at most 1. Because the current consumption takes on integer values, it would otherwise be greater or equal than 2 all the time, requiring speed $v \geq 2$. Hence, at $t_1$ at least for one of the directions the current consumption is 0. This can only happen if the fire burns along the back of a vertical barrier and has fully consumed all barriers previously touched in this direction. For example, such a 0-interval can be seen at $d_2$ in Figure 1.

W. l. o. g. assume this happens on the left-hand side. By Lemma 3.1, we can also assume that barriers are strictly increasing in length in both directions. Consider the time $t$ and total consumption $\mathcal{C}^l_{\mathcal{S}^*}(t)$ in the left direction at the beginning of a 0-interval.

$$t = \sum_{k=1}^{i} a_k + b_i + \max(0, 2b_{i-1} - b_i) \qquad \mathcal{C}^l_{\mathcal{S}^*}(t) = \sum_{k=1}^{i} a_k + \sum_{k=1}^{i} b_k - s$$

For an explanation of the $\max(0, 2b_{i-1} - b_i)$ summand, see Figure 2.

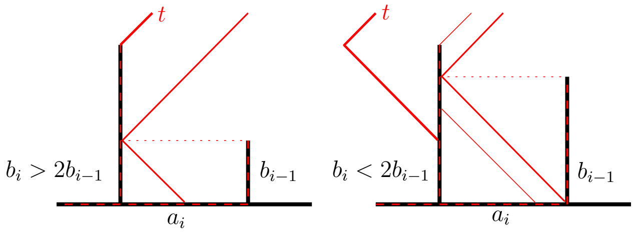


**Figure 2** The two possible configurations at the beginning of a 0-interval at time $t$. In the first case, the fire reached the lower end of the $b_i$ barrier first, so the 0-interval starts exactly when the upper end is reached. In the second case, the upper end was reached first, so the 0-interval begins when the lower end is reached after additional time $2b_{i-1} - b_i$.

Assume $\mathcal{Q}^l_{\mathcal{S}^*}(t) \leq 1$, then $\frac{\mathcal{C}^l_{\mathcal{S}^*}(t)}{t} \leq 1$ and $\mathcal{C}^l_{\mathcal{S}^*}(t) \leq t$

$$\begin{aligned} \Rightarrow \quad \sum_{k=1}^{i} a_k + \sum_{k=1}^{i} b_k - s &\leq \sum_{k=1}^{i} a_k + b_i + \max(0, 2b_{i-1} - b_i) \\ \Rightarrow \quad \sum_{k=1}^{i-2} b_k + b_{i-1} + b_i &\leq \max(b_i, 2b_{i-1}) + s \end{aligned}$$

Due to Lemma 3.1 $b_{i-1} + b_i > \max(b_i, 2b_{i-1})$ and $\sum_{k=1}^{i-2} b_k > (i-2) \cdot b_1 > s$ for $i$ larger than some threshold $i_0$. As the left-hand current consumption is 0 for arbitrarily large $t$, we can assume $i > i_0$. But this is a contradiction. Hence $\mathcal{Q}^l_{\mathcal{S}^*}(t) > 1$ at the beginnings of all 0-intervals after some threshold $t_0$ in the left-hand direction.

Hence for the whole barrier set $\mathcal{S}^*$ to be viable for $v < 1.5$, the consumption-ratio $\mathcal{Q}^r_{\mathcal{S}^*}(t)$ in the right-hand direction must be below 0.5 at exactly these moments. Hence for the right-hand direction 0-intervals exist for arbitrarily large $t$.

The ends of those 0-intervals are local minima of that directions consumption-ratio. Consider the time $t$ and total consumption $\mathcal{C}^r_{\mathcal{S}^*}(t)$ in the right-hand direction at the end of such an interval.

$$t = \sum_{k=1}^{i} c_k + d_i + \min(c_{i+1}, d_i) \qquad \mathcal{C}^r_{\mathcal{S}^*}(t) = \sum_{k=1}^{i} c_k + \sum_{k=1}^{i} d_k - s$$

For an explanation of the $\min(c_{i+1}, d_i)$ summand, see Figure 3.

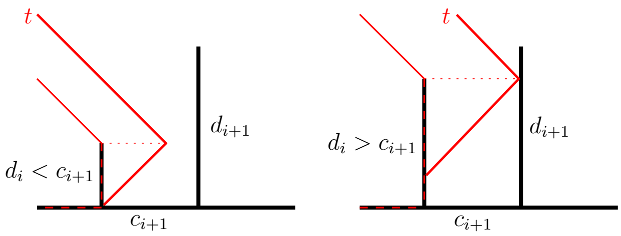


**Figure 3** The two possible configurations at the end of a 0-interval at time $t$. In the first case, the 0-interval ends, when the fire reaches the highway, $d_i$ time units after passing the top of the $d_i$ barrier. In the second case, the next barrier is reached before the highway, so the 0-interval ends, $c_{i+1}$ time units after passing the top of the $d_i$ barrier.

Assume $\mathcal{Q}^r_{\mathcal{S}^*}(t) < 0.5$, then $\frac{\mathcal{C}^r_{\mathcal{S}^*}(t)}{t} < 0.5$ and $2 \cdot \mathcal{C}^r_{\mathcal{S}^*}(t) < t$

$$\begin{aligned} \Rightarrow \quad 2 \cdot \sum_{k=1}^{i} c_k + 2 \cdot \sum_{k=1}^{i} d_k - 2s &< \sum_{k=1}^{i} c_k + d_i + \min(c_{i+1}, d_i) \\ \Rightarrow \quad \sum_{k=1}^{i} c_k + 2 \cdot \sum_{k=1}^{i} d_k &< 2d_i + 2s \Rightarrow \sum_{k=1}^{i} c_k + 2 \cdot \sum_{k=1}^{i-1} d_k < 2s \end{aligned}$$

As above $\sum_{k=1}^{i-1} d_k > (i-1)d_1 > 2s$ for $i$ larger than some threshold $i_0$. Hence after some threshold time $t_0$, even at the local minima the consumption-ratio of the right-hand direction is above 0.5, hence it is always above 0.5. Therefore $\mathcal{Q}_{\mathcal{S}^*}(t)$ exceeds 1.5 for arbitrarily large $t$, which makes $\mathcal{S}^*$ not viable for $v \leq 1.5$. ◀

## 4 A viable barrier set for $v = 1.8944\ldots$

Assume we want to construct a barrier set $\mathcal{S}$ improving on the naive strategy, which is viable for $v = 2$. The consumption-ratio only decreases when the current consumption is below the current ratio. As the current consumption is an integer value, this only happens when it is at most 1, which means we have a 0-interval in at least one direction. Hence we would want these intervals to be as long as possible, which results in the following conditions:

$$a_{i+1} \geq b_i \quad \wedge \quad b_{i+1} \geq 2b_i \qquad \forall i \geq 1$$
$$\text{equivalently } c_{i+1} \geq d_i \quad \wedge \quad d_{i+1} \geq 2d_i \qquad \forall i \geq 1$$

This forces the 0-intervals to always start exactly when the fire reaches the upper end of a vertical wall and end when the fire reaches the highway again after burning along the full length along the back of the already consumed vertical wall; see Figure 2 and 3. For a single direction this results in a repeating cycle of time intervals as seen in Figure 4. From our reasoning about the lower bound, we know that 0-intervals are necessary for both

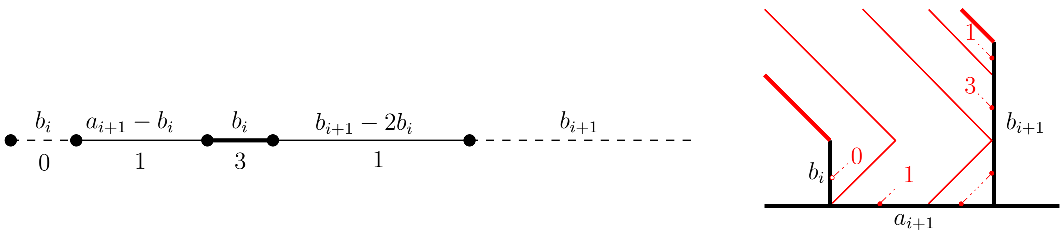


**Figure 4** A repeating time interval cycle. The values 0, 1 and 3 denote the current consumption in this direction.

directions in order to achieve $v < 2$. The idea is to construct the barrier set in such a way that the 0-intervals always appear in an alternating fashion, so the local maxima in the consumption-ratio of one direction can be countered by the 0-intervals of the other direction.

Consider the periodic interlacing of time intervals as illustrated in Figure 5, where the end point of the 0-interval from one direction coincides with the beginning of the 3-interval of the other direction.

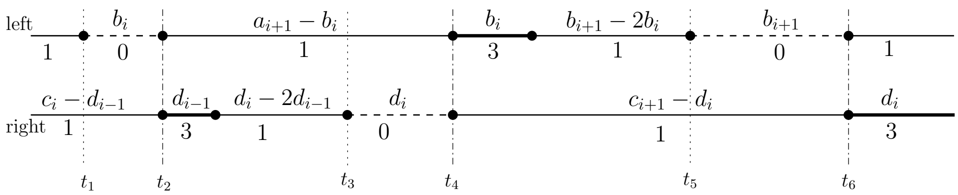


**Figure 5** The periodic interlacing of time intervals.

The current consumption is always greater than 1, since the 0-intervals do not overlap. As we are trying to construct a barrier set viable for some $v < 2$, $\mathcal{Q}_\mathcal{S}(t)$ must be smaller than 2 at all times. Then, $t_1$, $t_3$ and $t_5$ have to be local maxima, while $t_2$, $t_4$ and $t_6$ have to be local minima of $\mathcal{Q}_\mathcal{S}(t)$. The idea to ensure that the global maximum of $\mathcal{Q}_\mathcal{S}(t)$ stays below 2 is to make all local maxima attain the same value $\mu$ and hope that it stays below 2.

So let $\mu := \mathcal{Q}_\mathcal{S}(t_1)$. Then $\mathcal{Q}_\mathcal{S}(t_3) = \mu$ if and only if, between $t_1$ and $t_3$, the ratio of consumption over time interval length is also $\mu$. Thus,

$$\frac{b_i \cdot (0+1) + d_{i-1} \cdot (1+3) + (d_i - 2d_{i-1}) \cdot (1+1)}{b_i + d_{i-1} + d_i - 2d_{i-1}} = \mu$$
$$\Leftrightarrow \qquad b_i + 2d_i = \mu(b_i + d_i - d_{i-1})$$
$$\Leftrightarrow \qquad d_i = \tfrac{\mu-1}{2-\mu} b_i - \tfrac{\mu}{2-\mu} d_{i-1}\,.$$

Similarly, by applying the same reasoning to $t_3$ and $t_5$, we obtain a second recursive formula $b_{i+1} = \frac{\mu-1}{2-\mu}d_i - \frac{\mu}{2-\mu}b_i$. Together these recursions can be represented by a matrix and solved by applying standard techniques. The values for $a_i$ (and $c_i$) can be expressed as linear combinations of $d_i$ and $b_i$ – see the interval between $t_2$ and $t_4$ ($t_4$ and $t_6$) in Figure 5.

Analysing the zeros of the characteristic polynomial indicates that this recursion has real solutions for $b_i$ and $d_i$ as long as $\mu \geq (2+\sqrt{5})/\sqrt{5}$. For the limiting case $\mu = (2+\sqrt{5})/\sqrt{5} = 1.8944\ldots$, we get $b_i = \left(2+\sqrt{5}\right) d_{i-1} = \left(2+\sqrt{5}\right)^2 b_{i-1}$.

To prove the final theorem, it remains to find initial values to get the recursion started, while maintaining $\mathcal{Q}_{\mathcal{S}}(t) \leq \mu$. Suitable values are

$$\begin{aligned} b_i &:= s \cdot \left(2+\sqrt{5}\right)^{2i} & \mu &:= \tfrac{2+\sqrt{5}}{\sqrt{5}} \approx 1.8944 & c_1 &:= \tfrac{(\mu-2)d_1+2s+b_1}{2-\mu} \\ d_i &:= s \cdot \left(2+\sqrt{5}\right)^{2i+1} & a_1 &:= s & a_2 &:= 2d_1 + c_1 - a_1 - b_1, \end{aligned}$$

which results in intervals given in Figure 6. Note that all barrier lengths only depend on $s$.

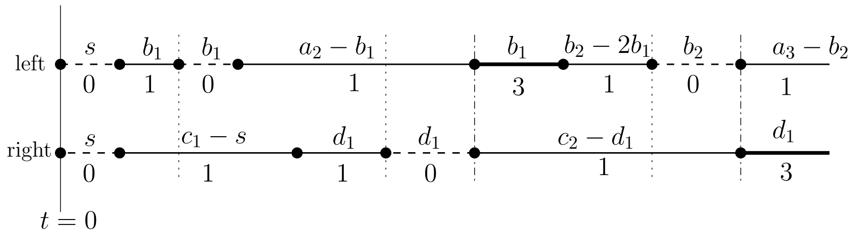


**Figure 6** Illustration of time intervals for a suitable starting situation.

▶ **Theorem 4.1.** *The highway can be protected with speed $v > \mu = \frac{2+\sqrt{5}}{\sqrt{5}} = 1.8944\ldots$*

## 5 Conclusion

We have shown the first non-trivial bounds for the highway protection problem. One future goal is to examine whether these results can be extended to more general barrier systems and/or the $L_2$ plane. We expect the analysis in the $L_2$ plane to be more difficult, especially finding the beginning and end times of the 0-intervals.

## References


1 P. Brass, K. D. Kim, H.-S. Na, and C.-S. Shin. Escaping offline searchers and isoperimetric theorems. *Computational Geometry*, 42(2):119 – 126, 2009.

2 A. Bressan. Differential inclusions and the control of forest fires. *Journal of Differential Equations*, 243(2):179 – 207, 2007.

3 A. Bressan, M. Burago, A. Friend, and J. Jou. Blocking strategies for a fire control problem. *Analysis and Applications*, 6(3):229–246, 2008.

4 S. Finbow and G. MacGillivray. The firefighter problem: A survey of results, directions and questions. Technical report, 2007.

5 F. V. Fomin, P. Heggernes, and E. J. van Leeuwen. The firefighter problem on graph classes. *Theor. Comput. Sci.*, 613(C):38–50, 2016.

6 R. Klein and E. Langetepe. Computational Geometry Column 63. *SIGACT News*, 47(2):34–39, 2016.

7 R. Klein, E. Langetepe, and C. Levcopoulos. A fire-fighter's problem. In *Proceedings 31st Symposium on Computational Geometry (SoCG'15)*, 2015.