

The Partition Spanning Forest Problem*

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Abstract

Given a set of colored points in the plane, we ask if there exists a crossing-free straight-line drawing of a spanning forest, such that every tree in the forest contains exactly the points of one color class. We show that the problem is NP-complete, even if every color class contains at most five points, but it is solvable in $O(n^2)$ time when each color class contains at most three points. If we require that the spanning forest is a linear forest, then the problem becomes NP-complete even if every color class contains at most four points.

1 Introduction

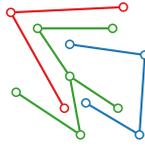
Let $P = \{p_1, \dots, p_n\}$ be a set of n points in the plane and let $C = \{C_1, \dots, C_k\}$ be a partition of P into k sets of points, called *color classes*, such that every point belongs to exactly one color class. We study the *partition spanning forest problem* which is defined as follows: Is there a crossing-free straight-line drawing of a spanning forest F that consists of k trees T_1, \dots, T_k such that each tree T_i , $1 \leq i \leq k$, contains exactly the points of the color class C_i ? Figure 1 shows an example with three color classes.

For $k = 1$, the problem is equivalent to finding a geometric spanning tree of P which trivially always exists. Hence, several optimization versions of this problem have been studied in the past; see Eppstein [4] for a survey. Bereg et al. [3] showed how to solve the problem in $O(n \log n)$ time in the case of $k = 2$. Hiu and Schaefer [5] proved that it is NP-complete to decide for two color classes $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ whether there exists an ordering π such that the geometric paths $a_{\pi_1}, \dots, a_{\pi_n}$ and $b_{\pi_1}, \dots, b_{\pi_n}$ are crossing-free. Bereg et al. [2] asked for not necessarily straight-line Steiner trees for each color class of minimum total length and gave a PTAS for $k = 2$ and a $(k + \varepsilon)$ -approximation for $k > 2$.

In this paper, we analyze the complexity of the partition spanning forest problem for color classes of bounded size. We give an $O(n^2)$ -time algorithm when each color class contains at most three points (Sec. 2) and show that the problem is NP-complete for up to five points per color class (Sec. 3); the complexity for four points remains open. In Section 4, we show that the *partition spanning linear forest problem*, where each tree is required to be a path, is NP-complete, even if every color class contains at most four points.

* This work started at the 14th European Research Week on Geometric Graph (GGWeek'17) in Vierhouten, The Netherlands. This research was funded in part by Humility & Conviction in Public Life, a project of the University Connecticut sponsored by the John Templeton Foundation.

34th European Workshop on Computational Geometry, Berlin, Germany, March 21–23, 2018.
This is an extended abstract of a presentation given at EuroCG'18. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.



■ **Figure 1** A solution to a problem instance with three color classes.

2 Color classes with at most three points

In the case where each color class of the input instance contains at most three points, the partition spanning forest problem can be solved in polynomial time. In fact, with this restriction the problem can be formulated as a 2-SAT problem.

Assume that our point set $P = \{p_1, \dots, p_n\}$ consists of n points. In the following we will understand the color classes as subsets $I \subseteq [n] := \{1, \dots, n\}$ of indices. For a point p_i we denote its color class by $I(p_i)$. We refer to the edges (p_i, p_j) where p_i and p_j are in the same color class as the *potential edges* of the instance. Observe that an arbitrary choice of the potential edges forms a solution to the problem (with at most three points per color class) if and only if it satisfies the following conditions: (i) For each point p_i , if $|I(p_i)| > 1$, then at least one potential edge incident to p_i must be chosen. (ii) For any pair of potential edges $p_i p_j$ and $p_k p_l$ that intersect in the interior, at most one of them is chosen. (iii) For any color class I with $|I| = 3$ one of the potential edges of that color is not chosen.

Observe that condition (iii) can be skipped, as any choice of potential edges satisfying conditions (i) and (ii) can be extended to also satisfy (iii).

We model the possible choices of potential edges that satisfy conditions (i) and (ii) by a 2-SAT formula as follows. For each potential edge (p_i, p_j) there is a variable x_{ij} with the interpretation that if x_{ij} is true, then the edge connecting p_i to p_j is *not* chosen as part of the solution, and otherwise it is.

Conditions (i) and (ii) can be expressed as 2-SAT formulas using the variables x_{ij} as follows. For condition (i), we create for each point p_i the (sub)formula $\bigvee_{j \in I(p_i) \setminus \{i\}} \neg x_{ij}$. Note that this is a 2-SAT formula since $|I(p_i) \setminus \{i\}| \leq 2$ by the assumption that each color class has size at most three. For any two potential edges (p_i, p_j) and (p_k, p_l) that cross, we add the clause $x_{ij} \vee x_{kl}$, thus enforcing condition (ii). It follows that the resulting 2-SAT formula φ is satisfiable if and only if the original instance of the partition spanning forest problem admits a solution. The formula has length at most $O(n^2)$ and can be constructed in $O(n^2)$ time as well. By using an efficient algorithm for 2-SAT [1], we get the desired algorithm. We summarize our construction in the following theorem.

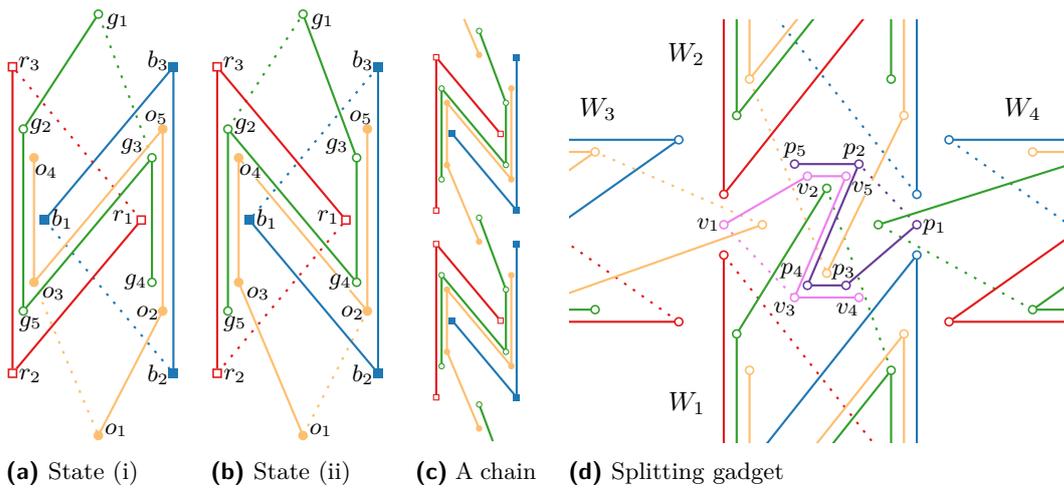
► **Theorem 1.** *The partition spanning forest problem for n points can be solved in $O(n^2)$ time if every color class contains at most three points.*

3 Color classes with at most five points

In this section we prove the following theorem:

► **Theorem 2.** *The partition spanning forest problem is NP-complete, even if every color class contains at most five points.*

The problem is obviously contained in NP. In order to show the NP-hardness, we perform a polynomial-time reduction from PLANAR 3-SATISFIABILITY. In this NP-hard [7] special case of 3SAT the input is a 3SAT formula φ whose variable–clause graph is planar. We can



■ **Figure 2** (a–c) The configurations of the wire gadget and (d) the splitting gadget.

assume that such a formula is given together with a contact representation \mathcal{R} of φ [6]. Thus, all variables are represented as horizontal line segments arranged on one line. Each clause c is represented as an E-shape turned by 90° such that the three vertical *legs* of the E-shape touch precisely the variables contained in c . For our reduction, we construct a set of colored points that admits a partition drawing if and only if φ is satisfiable.

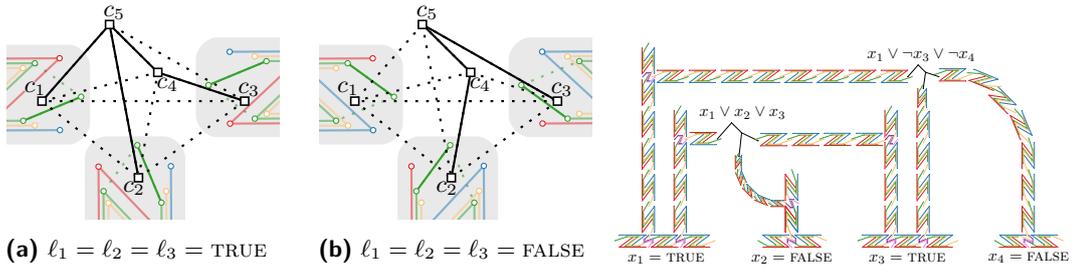
Overview. We introduce five types of gadgets. For each variable u we create a *variable gadget* which admits exactly two distinct partition drawings. These drawings correspond to the two truth states of u . *Wire gadgets* are used to propagate these states to the *clause gadgets*, one of which is created for every clause c . The clause gadget of c ensures that gadget configurations of the variables contained in c correspond to a truth assignment in which at least one of the literals of c is satisfied. In order to connect our gadgets appropriately we also require a *splitting gadget*, which splits one wire into two wires, and we require a gadget that flips the state transported along a wire. We proceed by describing our gadgets in detail. Note that different gadgets always use different color classes, even if we might give them the same name in the construction (so there are many *red* color classes in an instance).

The wire gadget. The wire gadget consists of four color classes; see Figure 2. The points of the *red* color class $R = \{r_1, r_2, r_3\}$ and the *blue* color class $B = \{b_1, b_2, b_3\}$ are arranged such that the convex hulls of R and B intersect in the two points $b_1b_2 \cap r_1r_2$ and $b_1b_3 \cap r_1r_3$. As a consequence, there are exactly two possible configurations for the red and blue spanning trees which can be used in a partition drawing, see Figure 2a and Figure 2b. Either choice uniquely determines the spanning tree of both the *green* color class $G = (g_1, \dots, g_5)$ and the *orange* color class $O = (o_1, \dots, o_5)$, as the edges of the red and blue spanning trees obstruct all other possible green and orange edges. Thus, there are exactly two possible partition drawings of the wire gadget. In particular, these two drawings satisfy the following.

► **Observation 3.** Any partition drawing of the wire gadget either contains (i) the edges g_1g_2 and o_1o_2 , but not the edges g_1g_3 and o_1o_3 , see Figure 2a; or (ii) the edges g_1g_3 and o_1o_3 , but not the edges g_1g_2 and o_1o_2 , see Figure 2b.

These two states (i) and (ii) may be propagated by creating *chains* of wire gadgets in which the convex hulls of consecutive gadgets intersect in two points as illustrated in Figure 2c. Consider two consecutive wire gadgets in a chain. By Observation 3, either both gadgets

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■ **Figure 3** The clause gadget between literals l_1, l_2, l_3 . ■ **Figure 4** A full example.

are in state (i) or both gadgets are in state (ii) due to the way their convex hulls intersect. As a consequence, the first gadget of the chain is in state (i) if and only if the last one is in state (i) as well. Chains are flexible structures and turns can easily be implemented by curving a chain. Further, the length of a chain may be adjusted by increasing or decreasing the distance between consecutive wire gadgets.

Splitting and inverting. The splitting gadget consists of two color classes $V = \{v_1, \dots, v_5\}$ (violet) and $P = \{p_1, \dots, p_5\}$ (purple) whose points are placed between two consecutive wires W_1, W_2 in a chain, see Figure 2d. The functionality of these two color classes is similar to the one of the color classes green and orange in the wire gadget: the state of W_1 and W_2 uniquely determines the spanning tree of both the violet and the purple color class. We may now attach one or two additional wires perpendicular to the chain such that their convex hulls intersect the convex hull of the splitting gadget, see W_3 and W_4 in Figure 2d. The edges incident to p_1 and v_1 in the purple and violet spanning trees allow precisely one state for both W_3 and W_4 .

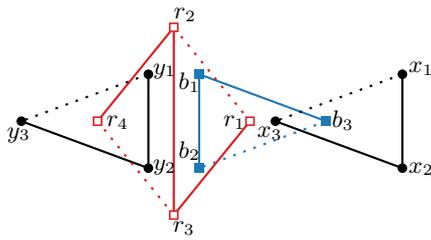
► **Observation 4.** *In any drawing of the splitting gadget, the state of the wires W_3 and W_4 differs from the state of W_1 and W_2 .*

In this sense, the splitting gadget does not only split a wire into two wires, it can also be used to flip the state propagated along a chain.

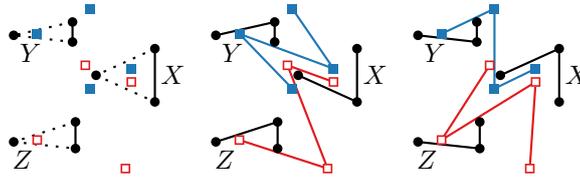
The variable gadget. The variable gadget is a horizontal chain to which we attach multiple wires using splitters. The number of wires attached from the top (bottom) matches the number of E-shape legs touching the variable from the top (bottom) in the contact representation \mathcal{R} of φ .

The clause gadget. The clause gadget for a clause of three literals l_1, l_2, l_3 consists of one color class with exactly five vertices c_1, \dots, c_5 . We place c_1, c_2 , and c_3 inside a wire gadget representing l_1, l_2 , and l_3 , respectively, and we place c_4 and c_5 between those as depicted in Figure 3. We will now show that the gadget is drawable if and only if at least one of l_1, l_2, l_3 is TRUE. In particular, we can always use an edge to connect c_4 and c_5 . We can connect c_3 to c_4 if l_3 is TRUE and we can connect c_3 to c_5 otherwise; similarly, we can connect c_2 to c_5 if $l_2 = \text{TRUE}$ and we can connect c_2 to c_4 otherwise. If $l_1 = \text{TRUE}$, then we can always connect c_1 to c_5 . However, if $l_1 = \text{FALSE}$, then we cannot connect c_1 to c_4 or c_5 , and we can connect it to c_2 or c_3 only if l_2 or l_3 is TRUE, respectively. Hence, the gadget is not drawable if $l_1 = l_2 = l_3 = \text{FALSE}$. Note that the connection from c_1 to c_3 might intersect the connection from c_2 to c_4 . However, we only have to use it if $l_1 = l_2 = \text{FALSE}$ and $l_3 = \text{TRUE}$; in this case, we can connect c_2 to c_3 instead of c_4 . Thus, the gadget is drawable if and only if at least one of l_1, l_2 , and l_3 is TRUE.

Layout and correctness. The wires that are attached to the variable gadgets are vertical and, by Observation 4, their state is inverted, so they propagate the negated variable. Hence,



■ **Figure 5** The wire gadget.



■ **Figure 6** The splitting gadget and its assignments.

if a literal is positive, we have to invert the state of the wire again. Two of the wires are supposed to enter the clause horizontally; for these two, if they correspond to a positive literal, we simply use another splitting gadget to make the wire horizontal. Otherwise, the wire makes a 90° degree turn to become horizontal and to propagate the negated variable. The third wire is supposed to enter the clause gadget vertically, so if its literal is negative, the vertical wire can directly connect to the clause. Otherwise, we use another splitting gadget followed by a 90° degree turn. See Figure 4 for an example of that shows all cases. Since the clause gadgets are drawable if and only if one of their literals is TRUE and since the wires propagate the states of the variable gadgets, the resulting instance is drawable if and only if the planar 3SAT formula φ is satisfiable, which proves the correctness of Theorem 2.

4 Linear forests for color classes with at most four points

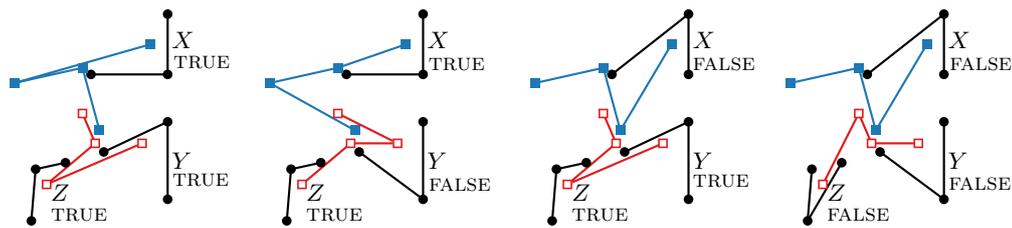
In this section we consider the additional restriction that the spanning forest is a linear forest, that is, each connected component is a path. Note that, if every color class contains at most three points, then every spanning forest is linear, so in this case we can solve the problem in polynomial time. On the other hand, we show that under this additional restriction, the problem is NP-complete already if every color class contains at most four points.

► **Theorem 5.** *The partition spanning linear forest problem is NP-complete, even if every color class contains at most four points.*

Again, the problem is clearly contained in NP. In order to show the NP-hardness, we again perform a polynomial-time reduction from PLANAR 3-SATISFIABILITY, but using different gadgets. As before, we construct a variable gadget, a splitting gadget, a wire gadget, and an inverter gadget. Instead of directly constructing a clause gadget, we will however construct an OR-gadget. The clause gadget can then be built by concatenating two OR-gadgets and enforcing the resulting variable gadget to be set to TRUE by crossing the appropriate edge with a new color class consisting of two points.

The variable, wire, and inverter gadgets. The variable gadget consists of one color class, the *black* color class $X = \{x_1, x_2, x_3\}$. Using a second color class, the *blue* color class $B = \{b_1, b_2, b_3\}$, we can enforce that the edge x_1x_2 must be drawn in any partition drawing. The classes B and X are placed in such a way that their convex hulls intersect in two points. In particular, there are two distinct partition drawings for B and X , corresponding to two truth states and x_1x_2 is present in both of them.

The wire gadget consists of four color classes, the *red* color class $R = \{r_1, r_2, r_3, r_4\}$ and the *blue* color class $B = \{b_1, b_2, b_3\}$, and two *black* color classes $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$, see Figure 5. Classes B and X are placed as in the variable gadget. Class Y is a copy of X , placed outside the convex hull of X and B . The point r_1 is placed inside the convex hull of B but outside the convex hull of X . The point r_4 is placed inside the convex



■ **Figure 7** Assignments of the OR-gadget.

hull of Y and r_2 and r_3 are placed such that the line through them separates the convex hulls of B and Y . Then, either partition drawing on X and B induces a unique partition drawing of R and Y , where the drawing on Y is the same as the drawing on X .

Placing Y as a copy of B instead of X , i.e., with only one point in the convex hull of R , we can also turn this gadget into an inverter gadget.

The splitting gadget. The splitting gadget consists of three variable gadgets X , Y , and Z , and two additional color classes, the *red* color class R and the *blue* color class B , see Figure 6. The truth assignment on X enforces some edges in R and B to be present, which then uniquely determines the partition drawing on the whole gadget. Note that the truth assignments on Y and Z are enforced as the negated truth assignment on X , so an additional inverter gadget might be needed depending on the required literal.

The OR-gadget. The OR-gadget consists of three variable gadgets X , Y , and Z , and two additional color classes, the *red* color class R and the *blue* color class B , see Figure 7. The truth assignments on X and Y enforce some edges in R and B to be present. It can be seen that the drawing of Z corresponding to the value TRUE can only be drawn if X or Y are also drawn corresponding to the value TRUE. In some of these cases, Z could also be drawn according to the value FALSE, but this does not affect the proof as it is still true that the constructed point set admits a partition drawing if and only if the planar 3SAT formula φ is satisfiable.

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