A Note on Flips in Diagonal Rectangulations*

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Abstract

Rectangulations are partitions of a square into axis-aligned rectangles. A number of results provide bijections between combinatorial equivalence classes of rectangulations and families of pattern-avoiding permutations. Other results deal with local changes involving a single edge of a rectangulation, referred to as flips, edge rotations, or edge pivoting. Such operations induce a graph on equivalence classes of rectangulations, related to so-called *flip graphs* on triangulations and other families of geometric partitions. In this note, we consider a family of flip operations on the equivalence classes of *diagonal rectangulations*, and their interpretation as transpositions in the associated *Baxter* permutations, avoiding the vincular patterns $\{3\underline{14}2, 2\underline{41}3\}$. This complements results by Law and Reading (JCTA, 2012) and provides a complete characterization of flip operations on diagonal rectangulations, in both geometric and combinatorial terms.

1 Introduction

In order to understand the underlying combinatorial structure of geometric space partitions such as triangle meshes or floorplans, it is often useful to define elementary operations that modify this structure locally. We can then connect distinct partitions using sequences of such operations. In triangulations, such a notion is known under the term of *flip*. A flip in a triangulation is typically defined as the replacement of an edge shared by two triangles forming a convex quadrilateral by the other diagonal of the quadrilateral. This allows the definition of a *flip graph*, the vertices of which are triangulations, and in which two triangulations are adjacent whenever one can be obtained from the other by a single flip. Flip graphs have applications in enumeration and random generation of geometric partitions as well as optimization, and have also been shown to have intimate links with many important structures in combinatorics, such as the Catalan objects, the Tamari lattice and the associahedra, cyclohedra, and partial cubes.

The objects of interest in this paper are *rectangulations*, defined as partitions of a square into axis-aligned rectangles. There exists a collection of results establishing bijections between classes of rectangulations and pattern-avoiding permutations [2, 4, 5, 8, 9]. A permutation σ is said to contain the *pattern* π, where π is another permutation, whenever there exists a subsequence of σ whose elements are in the same relative order as the elements of π. Pattern-avoiding permutations are families of permutations that do not contain any occurrence of one or more given patterns. We use the more general *vincular* notation for forbidden patterns, in which an underlined pair of elements indicates that they need to occur consecutively in the permutation . For instance, forbidding the pattern $3\underline{14}2$ amounts to forbidding all occurrences of the pattern 3142 with the added condition that 1 and 4 must occur consecutively.

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This is an extended abstract of a presentation given at EuroCG'18. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

Different types of local operations can be defined on rectangulations, which have been given different names, such as flips, local moves, edge rotations, or edge pivoting. In general, they all consist in replacing a horizontal edge of the rectangulation by a vertical one, or vice versa. In what follows, and with a slight abuse of terminology, we will refer to all those operations under the common name of *flip*.

Law and Reading [8] described a family of flips on rectangulations and provided an elegant combinatorial characterization. They showed that two rectangulations were connected by such a flip if and only if they were in the cover relation of a certain natural lattice structure, analogous to the Tamari lattice on triangulations. This lattice was also studied by Giraudo [7] under the name of *Baxter lattice*. Ackerman, Barequet and Pinter [3] defined related flip operations on rectangulations of a point set. These rectangulations are defined on a given point set so that every point lies on a segment of the rectangulation, and vice versa. Ackerman et al. studied the flip graph induced by these operations [1]. The flips considered by Ackerman et al. are the same as the ones in Law and Reading whenever the point set lies on the diagonal. Their results include a linear upper bound on the diameter of this flip graph (see [1], Section 4).

Our results. We first describe a known bijection from diagonal rectangulations to Baxter permutations, avoiding the vincular patterns $\{3\underline{14}2, 2\underline{41}3\}$. Then we consider flip operations on diagonal rectangulations, classify the different kinds of flips and give a combinatorial interpretation for each. Those involving edges that do not intersect the diagonal of the square, have already been characterized by Law and Reading [8]. For the others, we prove that the obtained flip graph is isomorphic to the graph on the corresponding Baxter permutations in which two Baxter permutations are adjacent whenever they differ by a single transposition of consecutive elements. This provides a complete one-to-one correspondence not only between rectangulations and Baxter permutations, but also between these sets of natural operations on the geometric and combinatorial structures. Due to space constraints, all proofs are omitted from this abstract, but can be found in the arXiv version.[1]

2 Diagonal rectangulations and Baxter permutations

The material of this section is adapted from Ackerman et al. [2], and Law and Reading [8]. A description of an essentially equivalent map in terms of pairs of twin binary trees was given by Felsner et al. [6].

A rectangulation is a partition of the unit square into axis-aligned rectangles. We define *vertices* as corners of the rectangles, and *edges* as line segments connecting two vertices, with no other vertex in between. The term *segment* is used to refer to inclusionwise maximal line segments of the rectangulation, possibly composed of several edges. We consider only rectangulations in which every vertex has exactly three incident edges, except the four vertices of the square, which have exactly two incident edges. We classify the vertices into four self-explanatory classes denoted by $\vdash$, $\dashv$, $\top$, and $\perp$.

We refer to the top-left to bottom-right diagonal of the square as the *main* diagonal. A *diagonal* rectangulation is a rectangulation in which every rectangle intersects the main diagonal. An

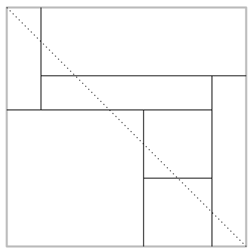


Figure 1 A diagonal rectangulation.

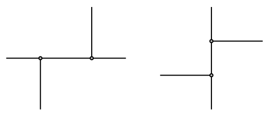


Figure 2 Forbidden configurations.

[1] https://arxiv.org/abs/1712.07919

example of diagonal rectangulation is given in Figure 1. However, we actually define diagonal rectangulations as equivalence classes of such partitions of the square, with respect to changes of vertex locations that preserve the adjacency relation between the rectangles. We have the following characterization of (the equivalence classes of) diagonal rectangulations.

▸ **Lemma 2.1.** *A rectangulation is diagonal if and only if it does not contain one of the two forbidden configurations of Figure 2.*

We can also consider the equivalence classes of rectangulations for which we can change the adjacency relation between the rectangles. Two rectangulations are then said to be equivalent when one can be obtained from the other by performing so-called *wall slides*, as shown on Figure 3. The equivalence relation is sometimes referred to as *R-equivalence* [4], and the *R*-equivalence classes are called *mosaic floorplans.*

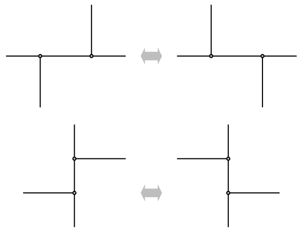


Figure 3 Wall slides.

▸ **Lemma 2.2.** *Every mosaic floorplan, or R-equivalence class, has a unique representative as a diagonal rectangulation.*

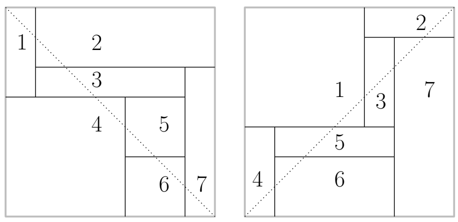


Figure 4 Illustration of the map B on the rectangulation R of Figure 1. The obtained Baxter permutation is $\mathrm{B}(R) = 4651372$.

We now describe the bijection B between diagonal rectangulations and *Baxter* permutations, which avoid the patterns $\{3\underline{14}2, 2\underline{41}3\}$. In order to define B, we define two linear orders on the rectangles of a rectangulation: the $\boxbslash$-order and the $\boxslash$-order. The $\boxbslash$-order is the order in which the rectangles are intersected by the main diagonal, from top-left to bottom-right. The $\boxslash$-order is obtained by taking the representative $\boxslash R$ of R in the equivalence class of mosaic floorplans such that the bottom-left to top-right diagonal intersects every rectangle. By Lemma 2.2, this representative exists and is unique. The order in which this diagonal intersects the rectangle is the $\boxslash$-order. The map B can then be described as follows:

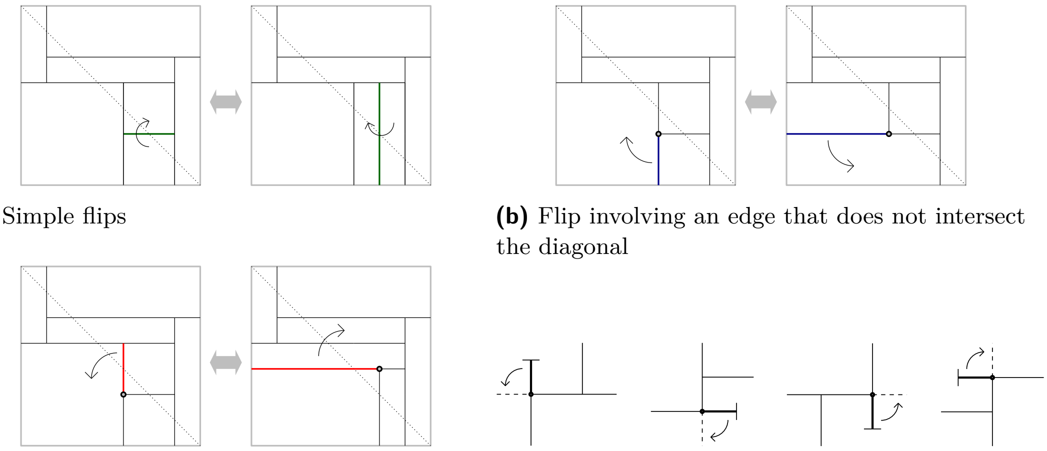
1. label the rectangles with respect to the $\boxbslash$-order,
2. enumerate the labels of the rectangles in the $\boxslash$-order.

▸ **Theorem 2.3.** [2] *The map* B *is a bijection between diagonal rectangulations with* n *rectangles and Baxter permutations on* n *elements.*

3 Flips

We consider only flipping edges that are not part of the boundary of the square. We say that an edge is *matched* at one of its endpoints whenever this endpoint is incident to another edge with the same (horizontal/vertical) orientation.

Simple flips involve edges cutting a rectangle into two rectangles, which are precisely the edges that are unmatched at both endpoints. In a diagonal rectangulation, all such edges must intersect the diagonal. A simple flip consists in replacing such a horizontal edge by a vertical one, or vice versa. When replacing the edge, we can always do it in such a way that the resulting rectangulation remains diagonal. An example of simple flip in the rectangulation of Figure 1 is given in Figure 5a.



(a) Simple flips

(b) Flip involving an edge that does not intersect the diagonal

(c) Flip involving an edge intersecting the diagonal

(d) The four types of unflippable edges matched at only one endpoint

Figure 5 Flips in diagonal rectangulations.

In some cases, an edge that is matched only at one of its endpoints can be *rotated* about this endpoint to yield another diagonal rectangulation. Examples of such flips are given in Figures 5b and 5c.

However, not all edges can be flipped. An edge is said to be unflippable in two cases. If the edge is matched at both endpoints, and if rotating this edge about any of the two endpoints yields a partition that is not a rectangulation. It can also be the case that an edge is matched at only one endpoint, and rotating it about this endpoint yields a rectangulation, but the obtained rectangulation is not diagonal. Unflippable edges matched at only one endpoint can be shown to come in four types, illustrated in Figure 5d.

4 A complete combinatorial characterization of flips

A *transposition* maps a permutation $\pi = \pi(1)\pi(2)\ldots\pi(j)\ldots\pi(k)\ldots\pi(n)$ to a permutation $\pi' = \pi(1)\pi(2)\ldots\pi(k)\ldots\pi(j)\ldots\pi(n)$. Furthermore, if the two values j and k satisfy $|\pi(j) - \pi(k)| = 1$, then the transposition is said to be a *transposition of consecutive elements*. If $k = j + 1$, then the transposition is said to be an *adjacent transposition*. Note that an adjacent transposition corresponds to a transposition of consecutive elements in the inverse permutation.

We first summarize a result of Law and Reading, characterizing some of the flip operations described above as a cover relation in a lattice, which can be found in Section 7 of [8]. In what follows, we will use the term *Law-Reading flips* to refer to those flips. In the original description (Section 7 of [8]), Law-Reading flippable edges are defined in terms of a *locking* operation. The following lemma gives a simple alternative definition of Law-Reading flips.

▸ **Lemma 4.1.** *Law-Reading flips are exactly the flips that are either simple, or that involve the rotation of a flippable edge that does not intersect the diagonal, as illustrated in Figure 5b.*

We now give a combinatorial characterization of Law-Reading flips proved in [8] using the map from rectangulations to Baxter permutations. Before stating the result, we must define the lattice dRec_n of diagonal rectangulations with n rectangles.

The *weak order* (also known as the weak *Bruhat* order) is a partial order on the set S_n of permutations of n elements in which a permutation π is smaller than another permutation

π' whenever the set of inversions of π is a subset of the set of inversions of π'. The cover relation of the weak order is the set of pairs of permutations that differ by a single adjacent transposition. The lattice dRec_n on diagonal rectangulations can be defined as the restriction of the weak order to the Baxter permutations corresponding to diagonal rectangulations with n rectangles. Recall that $B(R)$ is the Baxter permutation associated with the diagonal rectangulation R.

▶ **Theorem 4.2** (Law and Reading [8]). *Let R and R' be two diagonal rectangulations. Then R and R' are connected by a Law-Reading flip if and only if* $\mathrm{B}(R)$ *and* $\mathrm{B}(R')$ *are in a cover relation in* dRec_n.

This means that the two Baxter permutations corresponding to the pair of rectangulations are related by a monotone sequence of adjacent transpositions, and the intermediate permutations, if any, are not Baxter permutations.

We define *Barcelona flips* as those flips that involve a flippable edge intersecting the main diagonal. Barcelona flips are either simple flips, or flips involving the rotation of an edge intersecting the diagonal, as shown in Figure 5c.

▶ **Lemma 4.3.** *Let R and R' be two diagonal rectangulations that are connected by a Barcelona flip. Then $\boxslash R$ and $\boxslash R'$ are connected by a Law-Reading flip.*

The lemma is illustrated in Figure 6. Combining the above lemma with an observation on the way to obtain $\boxslash R$ from the inverse permutation $B(R)^{-1}$, and the characterization of Law-Reading flips in Theorem 4.2, we can already conclude that a Barcelona flip in a rectangulation R corresponds to a sequence of adjacent transpositions in $B(R)^{-1}$, that is, a sequence of transpositions of consecutive elements in $B(R)$. In fact, we can prove the following precise correspondence, involving only *single* transpositions.

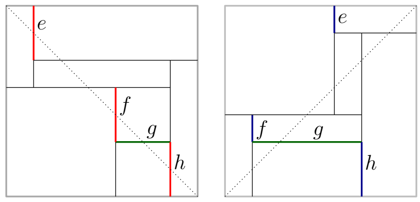


Figure 6 Illustration of Lemma 4.3: edges that can be flipped by a Barcelona flip in R (left) can be flipped by a Law-Reading flip in $\boxslash R$ (right).

▶ **Lemma 4.4.** *Let R and R' be two diagonal rectangulations. Then R and R' are connected by a Barcelona flip if and only if* $\mathrm{B}(R)$ *and* $\mathrm{B}(R')$ *differ by a single transposition of consecutive elements.*

The following theorem summarizes our results.

▶ **Theorem 4.5.** *Two diagonal rectangulations R and R' are connected by a flip if and only if one of these two conditions hold:*

- $\mathrm{B}(R)$ *and* $\mathrm{B}(R')$ *differ by a single transposition of consecutive elements,*
- $\mathrm{B}(R)$ *and* $\mathrm{B}(R')$ *are in a cover relation in* dRec_n.

Furthermore, R and R' are connected by a simple flip if and only if both conditions hold.

The flip graph on diagonal rectangulations with four rectangles is given in Figure 7.

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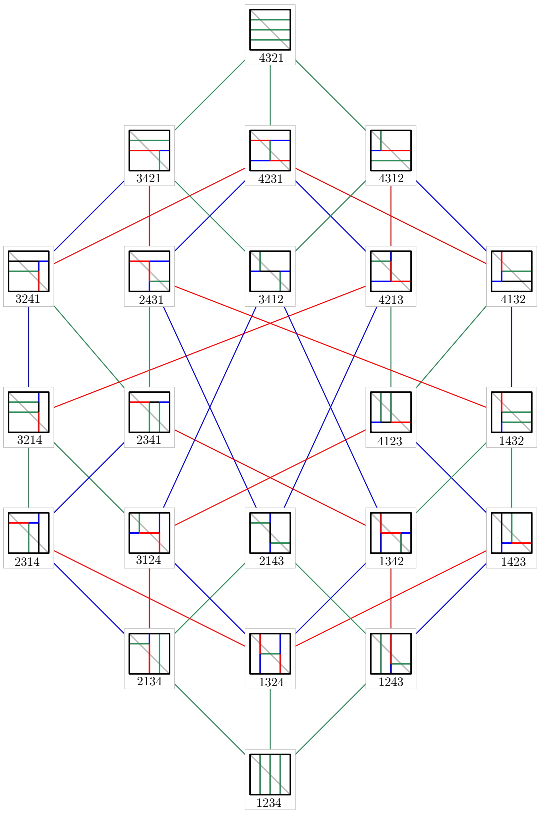

Figure 7 The flip graph on diagonal rectangulations made of four rectangles. In each rectangulation, the green edges are simply-flippable, and the blue and red edges are respectively Law-Reading and Barcelona-flippable, but not simply flippable.

References

1 Eyal Ackerman, Michelle M. Allen, Gill Barequet, Maarten Löffler, Joshua Mermelstein, Diane L. Souvaine, and Csaba D. Tòth. The Flip Diameter of Rectangulations and Convex Subdivisions. *Discrete Mathematics & Theoretical Computer Science*, 18(3), 2016.

2 Eyal Ackerman, Gill Barequet, and Ron Y. Pinter. A bijection between permutations and floorplans, and its applications. *Disc. App. Math.*, 154(12):1674–1684, 2006.

3 Eyal Ackerman, Gill Barequet, and Ron Y. Pinter. On the number of rectangulations of a planar point set. *J. Comb. Theory, Ser. A*, 113(6):1072–1091, 2006.

4 Andrei Asinowski, Gill Barequet, Mireille Bousquet-Mélou, Toufik Mansour, and Ron Y. Pinter. Orders induced by segments in floorplans and (2-14-3, 3-41-2)-avoiding permutations. *Electr. J. Comb.*, 20(2):P35, 2013.

5 Serge Dulucq and Olivier Guibert. Stack words, standard tableaux and Baxter permutations. *Disc. Math.*, 157(1):91–106, 1996.

6 Stefan Felsner, Éric Fusy, Marc Noy, and David Orden. Bijections for Baxter families and related objects. *J. Comb. Theory, Ser. A*, 118:993–1020, 2011.

7 Samuele Giraudo. Algebraic and combinatorial structures on pairs of twin binary trees. *J. Algebra*, 360(Supplement C):115–157, 2012.

8 Shirley Law and Nathan Reading. The Hopf algebra of diagonal rectangulations. *J. Comb. Theory, Ser. A*, 119(3):788–824, 2012.

9 Nathan Reading. Generic rectangulations. *Eur. J. Comb.*, 33(4):610–623, 2012.