Reconstructing a convex polygon from its ω-cloud *

Prosenjit Bose[1], Jean-Lou De Carufel[2], Elena Khramtcova[3], and Sander Verdonschot[4]

1 **Carleton University, Ottawa, Canada**
`jit@scs.carleton.ca`
2 **University of Ottawa, Ottawa, Canada**
`jdecaruf@uottawa.ca`
3 **Université libre de Bruxelles (ULB), Brussels, Belgium**
`elena.khramtsova@gmail.com`
4 **Carleton University, Ottawa, Canada**
`sander@cg.scs.carleton.ca`

Abstract

An ω-wedge is the set of all points contained between two rays emanating from a single point (the apex) and separated by an angle $\omega < \pi$. Given a convex polygon P, we place the ω-wedge so that it contains P and its both rays are tangent to P. The ω-cloud of P is the curve traced by the apex of the ω-wedge as it rotates around P while maintaining tangency in both rays.

We investigate reconstructing a polygon P from its ω-cloud. Previous work on reconstructing P from probes with the ω-wedge required knowledge of the points of tangency between P and the two rays of the ω-wedge. Here we show that if ω is known, the ω-cloud alone uniquely determines P, and we give a linear-time reconstruction algorithm. Furthermore, even if we only know that $\omega < \pi/2$, we can still reconstruct P, albeit in cubic time in the number of vertices. This reduces to quadratic time if in addition we are given the location of one of the vertices of P.

1 Introduction

"Geometric probing considers problems of determining a geometric structure or some aspect of that structure from the results of a mathematical or physical measuring device, a probe." [6, Page 1] Many probing tools have been studied in the literature such as finger probes, hyperplane (or line) probes, diameter probes [5], x-ray probes, histogram (or parallel x-ray) probes, half-plane probes and composite probes to name a few. See the review of Skiena [6] and for more recent results, see Bose et al. [1] and references therein.

Closely related to a geometric probing problem is a *reconstruction* problem: Can one reconstruct an object given a set of probes? Surprisingly, for diameter probes this is not the case [5]. An ω-wedge, introduced by Bose et al. [1], is a probing device that is the (closed) set of all points contained between two rays emanating from a single point called the *apex* of the wedge. The angle ω formed by the two rays is such that $0 < \omega < \pi$. A probe of a convex n-gon P is *valid* when P is inside the wedge and both rays of the wedge are tangent to P, see Fig. 1a. A valid probe returns the coordinates of the apex and of the two points of contact between the rays and the polygon. A convex n-gon can be reconstructed using between $2n - 3$ and $2n + 5$ such probes [1], depending on the value of ω and the number of *narrow vertices* (vertices whose internal angle is at most ω) in P. As the ω-wedge rotates around P, the locus of the apex of the ω-wedge describes a curve called an *ω-cloud* (see Fig. 1c).

The ω-cloud is a generalization of the diameter function of Rao and Goldberg [5]. A *diameter probe* consists of two parallel calipers turning around a convex object P in the plane. The *diameter function* returns the distance between the calipers as they turn around P. As

* P. Bose, J.-L. De Carufel, and S. Verdonschot were partially supported by NSERC, E. Khramtcova was partially supported by SNF Early Postdoc Mobility grant P2TIP2-168563 and F.R.S.- FNRS.

34th European Workshop on Computational Geometry, Berlin, Germany, March 21–23, 2018.
This is an extended abstract of a presentation given at EuroCG'18. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

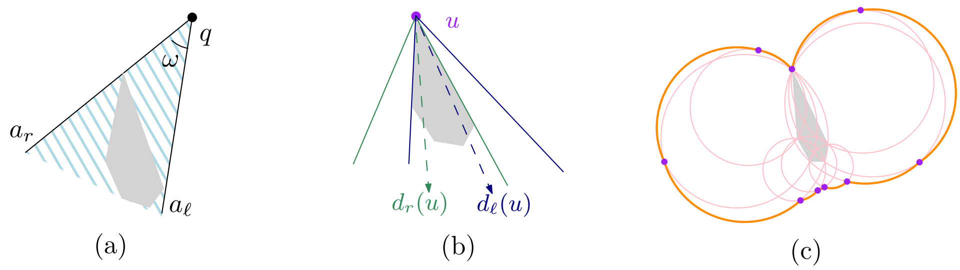


Figure 1 A convex polygon P (shaded area), and: (a) A minimal ω-wedge of P (tiling pattern); (b) A narrow vertex u of P, wedges $W_\ell(u)$ and $W_r(u)$ (bounded by, resp., blue and green solid lines) and their directions $d_\ell(u)$ and $d_r(r)$ (dashed lines); (c) The ω-cloud Ω of P: the arcs (orange lines), pivots (purple disk marks), and all the supporting circles (light-pink lines).

two different convex polygons can have the same diameter function [5], recovering a convex n-gon given only its diameter function is not always possible. An ω-wedge can be seen as two non-parallel calipers turning around P. Here we show that the ω-cloud function is free from the above drawback, and thus is a more advantageous than the diameter function.

In this paper, we analyze the structure of ω-cloud, resulting in many interesting properties, including the uniqueness of the polygon for a given ω-cloud (see Sec. 2). Further, we show, that if the value of ω is known, P can be reconstructed from its ω-cloud in $O(n)$ time and $O(k)$ space, where k is the number of narrow vertices; the required space is constant for any fixed value of ω (see Sec. 3.1). If the value of ω is not known, we can still recover P, as long as $\omega < \pi/2$. In this case, we give an $O(n^3)$ time and $O(n^2)$ space reconstruction algorithm. The time complexity reduces to $O(n^2)$ if, in addition, we know a vertex of P and no three vertices of P are on one supporting circle of an arc of the ω-cloud (see Sec. 3.2). Due to space constraints, many proofs are omitted; they can be found in the full version of this paper [2].

2 Properties of the ω-cloud

In this section we introduce the necessary definitions and notation, and then we list the properties of the ω-cloud (Lemmas 2.2-2.6), which lead to the uniqueness of the polygon for a given ω-cloud (Thm. 2.8) and are the basis for our reconstruction algorithms (see Sec. 3).

Let P be an n-vertex convex polygon in $\mathbb{R}^2$. For any vertex v of P, let $\alpha(v)$ be the internal angle of P at v. Let ω be an angle with $0 < \omega < \pi$. Consider an ω-*wedge* W; recall that it is the set of points contained between two rays emanating from the same point q (the apex of W) such that the angle between the two rays is ω. We call the ray a_ℓ (resp., a_r) that bounds W from the left (resp., right) as seen from q, the *left* (resp., *right*) *arm* of W. See Fig. 1a. We say that an ω-wedge W is *minimal* for P if P is contained in W and the arms of W are tangent to P. The *direction* of W is given by the bisector ray of the two arms of W. For each direction, there is a unique minimal ω-wedge.

▶ **Definition 2.1.** The ω-*cloud* of P is the locus of the apexes of all minimal ω-wedges for P.

The ω-cloud Ω of P is a circular sequence of circular arcs, where each two consecutive arcs share an endpoint. An *arc* Γ of the ω-cloud is a maximal contiguous portion of Ω that corresponds to apexes of combinatorially same ω-wedges (i.e., the arms of all the wedges touch the same pair of vertices in P). Each two consecutive arcs share an endpoint called *pivot* of Ω. If $\omega \geq \pi/2$, two consecutive arcs of the ω-cloud can have same supporting circle. We call the pivot connecting such arcs a *hidden pivot*. There are between n and $2n$ pivots [3].

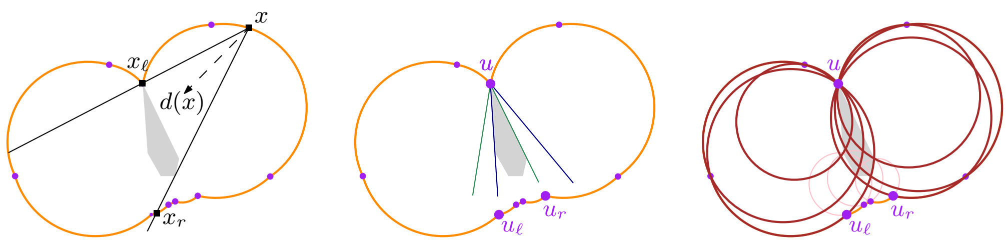
A vertex v of P is *narrow* if $\alpha(v) \leq \omega$. A pivot of Ω coincides with a vertex of P if and only if that vertex is narrow; such a pivot is also *narrow*. If $\alpha(v) < \omega$, we call v (the vertex or the pivot) *strictly narrow*. As the *portion of* Ω *between two points* $s, t \in \Omega$, denoted by Ω_{st}, we refer to the open portion of Ω encountered when traversing Ω from s to t clockwise. The *angular measure* of an arc Γ is the angle spanned by Γ, measured from the center of its supporting circle. For two points s, t on Ω, the *total angular measure* of Ω from s to t, denoted by $D_\Omega(s,t)$, is the sum of the angular measures of all arcs in Ω_{st}.

Each point x in the interior of an arc corresponds to a unique minimal ω-wedge $W(x)$ with direction $d(x)$. Let u be a pivot of Ω. If u is not strictly narrow, u also corresponds to a unique minimal ω-wedge $W(u)$ with direction $d(u)$. Otherwise, u corresponds to a closed interval of directions $[d_\ell(u), d_r(u)]$, where the angle between $d_\ell(u)$ and $d_r(u)$ equals $\omega - \alpha(u)$. See Fig. 1b. Let $W_\ell(u)$ and $W_r(u)$ denote the minimal ω-wedges with apex at u and directions resp. $d_\ell(u)$ and $d_r(u)$. For points x on Ω that are not strictly narrow pivots, we define $d_r(x)$ and $d_\ell(x)$ both to be equal to $d(x)$, and both $W_\ell(x), W_r(x)$ equal to $W(x)$.

The following is a crucial property of the ω-cloud, lying in the basis of the other properties.

▶ **Lemma 2.2.** *Let s and t be two points on Ω such that there are no narrow pivots between s and t. Then the angle β between $d_r(s)$ and $d_\ell(t)$ is $D_\Omega(s,t)/2$.*

Proof (sketch). If Ω_{st} is a single arc, angle β equals the angle between the left arms of the two minimal ω-wedges corresponding to $d_r(s)$ and $d_\ell(t)$. This angle by elementary geometry equals $D_\Omega(s,t)/2$. If Ω_{st} consists of several arcs, since none of the pivots between s and t narrow, angle β is the total sum of the corresponding angles for all the arcs.



(a) (b) (c)

Figure 2 (a) Point x in the interior of an arc of Ω, wedge $W(x)$, direction $d(x)$, and points x_ℓ and x_r. (b) Narrow pivot u, wedges $W_\ell(u)$ and $W_r(u)$, points u_ℓ and u_r. (c) Narrow pivot u, the points $v = u_\ell$ and $w = u_r$, and the supporting circles of all the arcs between them (bold brown lines).

▶ **Corollary 2.3.** *For any arc Γ of Ω, $|\Gamma| \leq 2(\pi - \omega)$.*

Let x be a point on Ω. The open ray of the right arm of $W_\ell(x)$ intersects Ω at least once. Among the points of this intersection, let x_ℓ be the one closest to x. Define the point x_r analogously for the left arm of $W_r(x)$. See Figs. 2a,b.

▶ **Lemma 2.4.** *(a) Both $\Omega_{x_\ell x}$, $\Omega_{x x_r}$ contain no narrow pivots. (b) If x is a narrow pivot, then $D_\Omega(x_\ell, x) = D_\Omega(x, x_r) = 2(\pi - \omega)$. (c) If x is not narrow, then either $D_\Omega(x, x_r) = 2(\pi - \omega)$, or x_r is the clockwise first narrow pivot after x. A symmetric statement holds for x_ℓ.*

▶ **Lemma 2.5.** *Let u be a pivot of Ω, and let v and w be the points on Ω such that $D_\Omega(v,u) = D_\Omega(u,w) = 2(\pi - \omega)$. (a) If pivot u is narrow, then the supporting circles of all the arcs of Ω_{vw} pass through u. See Fig. 2c. (b) If Ω_{vu} consists of a single arc, or there is an arc Γ of Ω_{vu} that is not incident to u, such that the supporting circle of Γ contains u, then u is narrow. A symmetric statement holds for Ω_{uw}.*

With Lemma 2.5 we can identify all narrow pivots on Ω that are not hidden, so we now turn our attention to the properties of hidden pivots.

▶ **Lemma 2.6.** *Let u be a hidden pivot of Ω, let Γ_ℓ and Γ_r be the two arcs of Ω incident to u, and let v and w be the other endpoints of Γ_ℓ and Γ_r, respectively. Then v, u, and w are all narrow and each of the arcs Γ_ℓ, Γ_r has angular measure $2(\pi - \omega)$.*

▶ **Corollary 2.7.** *If all arcs of the ω-cloud of P have the same supporting circle C, then $k = \pi/(\pi - \omega)$ is an integer and P is a regular k-gon inscribed in C.*

Suppose Ω is the ω-cloud of a convex polygon P. Lemmas 2.5 and 2.6 uniquely identify the narrow pivots of Ω, which are the narrow vertices of P (including hidden narrow pivots). By Lemma 2.4a, the portion of Ω between any two narrow pivots has total angular measure at least $2(\pi - \omega)$. Thus the components of P as defined by excluding all narrow vertices are uniquely determined by Ω: For each such component Lemma 2.4b,c gives the minimal ω-wedge W with the apex at some point x in that component. Wedge W intersects the supporting circle of an arc Γ, incident to or containing x, in the two vertices of P tangent to the arms of the minimal ω-wedge as its apex traverses Γ. This proves the following.

▶ **Theorem 2.8.** *Given an angle ω, and a circular sequence Ω of at least two circular arcs, there is at most one convex polygon P such that Ω is the ω-cloud of P.*

3 Reconstructing P from its ω-cloud

Let Ω be a circular sequence of circular arcs. Our goal is to reconstruct the convex polygon P for which Ω is the ω-cloud, or determine that no such polygon exists. Sec. 3.1 and 3.2 consider respectively ω to be given or not. As opposed to the above sections, here we consider arcs of Ω to be *maximal* portions of the same circle, that is, no two neighboring arcs have the same supporting circle. This is natural for the reconstruction task, since as an input we are given a locus of the apexes of all the minimal ω-wedges and no additional information.

If Ω is a single (maximal) arc, i.e., it is a circle C, then P is not unique: By Cor. 2.7, it is a regular $\pi/(\pi - \omega)$-gon inscribed in C; but the position of its vertices on C is impossible to identify given only Ω and ω. Thus we assume that Ω has at least two arcs.

3.1 An ω-aware reconstruction algorithm

We are given an angle ω, $0 < \omega < \pi$, and a circular sequence of at least two circular arcs Ω. We want to check if Ω is the ω-cloud of some convex polygon P, and to return P if this is the case. Our algorithm performs two passes through Ω. The first pass computes a list S of all strictly narrow vertices of P that are not hidden pivots. With each such vertex u, we store the supporting lines of the two edges of P incident to u. The second pass reconstructs separately the portion of P for each connected component of Ω induced by the vertices in S. After giving the procedure for the latter task in Lemma 3.1, we present the two passes.

For two points u and v on Ω, let P_{uv} be the union of the edges and vertices of P touched by the arms of the minimal ω-wedge as its apex traverses Ω_{uv}. Note that P_{uv} consists of at most two connected portions of P; it is possible that one of the portions is a single vertex.

▶ **Lemma 3.1.** *Given a portion Ω_{uv} with no strictly narrow pivots and direction $d_r(u)$, the portion P_{uv} can be reconstructed in time linear in the number of arcs in Ω_{uv} and $O(1)$ space.*

Proof. Let $\Gamma = uu'$ be the arc of Ω_{uv} incident to u, and let C be the supporting circle of Γ. See Fig. 3a. The values of ω and $d_r(u)$ determine the wedge $W_r(u)$. The intersection

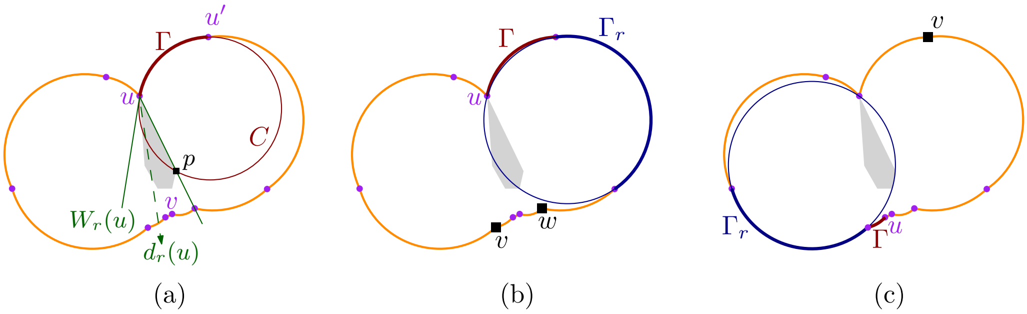


Figure 3 (a) Illustration for the proof of Lemma 3.1. First pass of the ω-aware algorithm: (b) u is a strictly narrow pivot, and (c) u is not a narrow pivot.

between $W_r(u)$ and C gives the two vertices of P touched by the minimal ω-wedges with apex on Γ. See the points u, p in Fig. 3a. Direction $d_\ell(u')$ equals $d_r(u) + D_\Omega(u, u')/2$ due to Lemma 2.2. If u' is inside Ω_{uv}, then u' is not a strictly narrow vertex, and thus the minimal ω-wedge with apex at u' is unique. This way we find the pair of vertices of P corresponding to each arc of Ω_{uv}. By visiting the pivots of Ω_{uv} in order, we find the vertices of each of the two chains of P_{uv} ordered clockwise. To avoid double-reporting vertices of P_{uv}, we keep the startpoints of the two chains, and if one chain reaches the startpoint of the other one, we stop reporting the points of the former one. This procedure visits each pivot of Ω_{uv} once, performing $O(1)$ operations at each pivot. Only $O(1)$ storage is required. ◀

First pass. We iterate through the pivots of Ω. For the currently processed pivot u, we maintain the point v on Ω such that $D_\Omega(v, u) = 2(\pi - \omega)$. If pivot u is narrow, we jump to the point on Ω at the distance $2(\pi - \omega)$ from u. Moreover, if u is strictly narrow, we add u to the list S. If u is not narrow, we process the next pivot of Ω. We now give the details.

Let Γ be the arc of Ω incident to u and following it. Let Γ_r be the arc following Γ, and C_r be the supporting circle of Γ_r. We consider cases depending on the angular measure $|\Gamma|$ of Γ:

(a) $|\Gamma| < 2(\pi - \omega)$. See Fig. 3b,c.
  (i) Circle C_r passes through u (see Fig. 3b). Then u is narrow by Lemma 2.5b. By tracing Ω, find the point w on it with $D_\Omega(u, w) = 2(\pi - \omega)$. Add u to the list S with the lines through vu and uw, if u is strictly narrow ($\angle vuw < \omega$). Set $v := u$, and $u := w$ (regardless the later condition).
  (ii) Circle C_r does not pass through u (see Figure 3b). Then u is not narrow by Lemma 2.5a. Set u to be the other endpoint of Γ, and update v accordingly.

(b) $|\Gamma| = 2(\pi - \omega)$. Then u is narrow by Lemma 2.5b. Let w be the other endpoint of Γ. Update S, v, and u as in item a(i).

(c) $|\Gamma| = 2t(\pi - \omega)$ for some integer $t > 1$. Then Γ is in fact multiple arcs separated by hidden pivots, see Lemma 2.6 and Corollary 2.3. Let p be the other endpoint of Γ. Let w and w' be the points on Γ such that $D_\Omega(u, w) = 2(\pi - \omega)$ and $D_\Omega(w', p) = 2(\pi - \omega)$. Update S, v, and u as in item a(i).

(d) Otherwise, stop and report that Ω is not an ω-cloud of any polygon.

Second pass. If list S is empty, we apply the procedure of Lemma 3.1 to the whole Ω. In particular, as both the start and the endpoint, we take the point x with which we completed the first pass of the algorithm; the point x' such that $D_\Omega(x', x) = 2(\pi - \omega)$ is already known from the first pass. Then $d_r(x) = d(x)$ is the direction of the minimal ω-wedge with the apex at x and the right arm passing through x'.

Suppose now the list S contains k vertices. They subdivide Ω into k connected portions that are free from strictly narrow non-hidden pivots. Each portion is treated as follows. If it is a single maximal arc of measure $2t(\pi - \omega)$, we separate it by $t-1$ equidistant points, and those points are exactly the vertices of the saught portion of P, see Lemma 2.6. Otherwise, it is a portion free from any strictly narrow pivots. We process it as in Lemma 3.1.

▶ **Theorem 3.2.** *Given an angle ω such that $0 < \omega < \pi$, and a circular sequence of circular arcs Ω which is not a single circle, there is an algorithm to check if Ω is the sequence of the maximal arcs, corresponding to the ω-cloud of some n-vertex convex polygon P, and to return P if this is the case. The algorithm works in $O(n)$ time, making two passes through the input, and it requires $O(k)$ storage, where k is the number of strictly narrow vertices of P. In particular, the required storage is constant for any fixed value of ω.*

3.2 An ω-oblivious reconstruction algorithm

▶ **Theorem 3.3.** *Given a circular sequence of circular arcs Ω, there is an algorithm that finds the convex polygon P such that Ω is the ω-cloud of P for some angle ω with $0 < \omega < \pi/2$, if such a polygon exists. Otherwise, it reports that such a polygon does not exist.*

(i) If no additional information is given, the algorithm works in $O(n^3)$ time and $O(n^2)$ space.

(ii) If a vertex v of P is given, and each supporting circle of an arc of Ω is guaranteed to pass through exactly two vertices of P, the algorithm works in $O(n^2)$ time and $O(n^2)$ space.

Proof. Our algorithm, summarized below, is based on the following property: if $0 < \omega < \pi/2$, each vertex of P lies on at least two distinct supporting circles of arcs of the ω-cloud of P.

In both cases (i) and (ii), we first construct the arrangement $\mathcal{A}$ of all the supporting circles of the arcs of Ω. This can be done in $O(n^2)$ time and $O(n^2)$ space [4].

(i) For each pair of vertices u, v of $\mathcal{A}$ incident to the same circle C, we do the following. Construct a wedge W passing through u and v, such that the apex x of W lies in the interior of the unique arc Γ of Ω supported by C, such that the corresponding angle ω at x is less than $\pi/2$. Run the algorithm of Thm. 3.2 for Ω, angle ω, and the direction $d(x)$ of W.

We process $O(n^2)$ pairs of vertices in total, processing one pair takes $O(n)$ time, thus the total time spent on the reconstruction of P is $O(n^3)$.

(ii) If v is not a vertex of $\mathcal{A}$, stop and return a negative answer. Otherwise, choose a circle C containing v. Run the above procedure for v and each vertex u of $\mathcal{A}$ with $u \in C, u \neq v$.

Since each circle of $\mathcal{A}$ incident to v is incident to only one other vertex of P, the minimal ω-wedge corresponding to Γ must pass through v. Thus we just consider one circle C incident to v. Since there are $O(n)$ vertices of $\mathcal{A}$ on the circle C, the algorithm runs in $O(n^2)$ time. ◀

References

1. P. Bose, J.-L. De Carufel, A. Shaikhet, and M. Smid. Probing convex polygons with a wedge. *Comput. Geom.*, 58:34–59, 2016. doi:10.1016/j.comgeo.2016.06.001.
2. P. Bose, J.-L. De Carufel, E. Khramtcova, and S. Verdonshot. Reconstructing a convex polygon from its ω-cloud. *ArXiv e-prints*, 2018. arXiv:1801.02162.
3. P. Bose, M. Mora, C. Seara, and S. Sethia. On computing enclosing isosceles triangles and related problems. *Int. J. of Comput. Geom. & Appl.*, 21(01):25–45, 2011.
4. H. Edelsbrunner, J. O'Rourke, and R. Seidel. Constructing arrangements of lines and hyperplanes with applications. *SIAM J. Computing*, 15(2):341–363, 1986.
5. A. S. Rao and K. Y. Goldberg. Shape from diameter: Recognizing polygonal parts with a parallel-jaw gripper. *I. J. Robotic Res.*, 13(1):16–37, 1994.
6. S. Skiena. Problems in geometric probing. *Algorithmica*, 4(4):599–605, 1989.