Deletion in abstract Voronoi diagrams in expected linear time*

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Abstract

Updating an abstract Voronoi diagram in linear time, after deletion of one site, has been an open problem for a long time. Similarly for concrete Voronoi diagrams of generalized sites, other than points. In this abstract we present a simple, expected linear-time algorithm for this task. We introduce the concept of a *Voronoi-like diagram*, a relaxed version of a Voronoi construct, that has a structure similar to an abstract Voronoi diagram without however being one. Voronoi-like diagrams serve as intermediate structures, which are considerably simpler to compute, thus, making an expected linear-time construction possible.

1 Introduction

The Voronoi diagram of a set S of n simple geometric objects, called sites, is a well-known geometric partitioning structure revealing proximity information for the input sites. *Abstract Voronoi diagrams* [7] offer a unifying framework to various concrete instances. Some classic Voronoi diagrams have been well investigated and optimal construction algorithms exist in many cases, see [2] for references and more information.

For certain *tree-like* Voronoi diagrams linear-time construction algorithms are well known to exist, e.g., [1, 4, 9, 5]. The first technique was introduced by Aggarwal et al. [1] for the Voronoi diagram of points in convex position, given their convex hull. It can be used to derive linear-time construction algorithms for other fundamental problems such as updating a Voronoi diagram of point-sites in linear time, after deleting one site. A much simpler randomized approach for the same problem has been introduced by Chew [4]. Klein and Lingas [9] adapted the linear-time framework of [1] to abstract Voronoi diagrams under restrictions, and showed that a *Hamiltonian abstract Voronoi diagram* can be computed in linear time, given the order of Voronoi regions along an unbounded simple curve, which visits each region *exactly once* and can intersect each bisector only *once*. This construction has been extended recently to include forest structures [3] under similar conditions, where no region can have multiple faces within the domain enclosed by a curve. The medial axis of a simple polygon is another well-known problem to admit a linear-time construction [5].

In this abstract we consider the problem of updating an abstract Voronoi diagram after deletion of one site and provide an expected linear-time algorithm to achieve this task. To the best of our knowledge, no linear-time construction algorithms are known for concrete diagrams of non-point sites, nor for abstract Voronoi diagrams. Related is our expected linear-time algorithm for the farthest-segment Voronoi diagram [6]. The approach in [6], however, is geometric, relying on star-shapeness and visibility properties of segment Voronoi regions that do not extend to the abstract model. In this abstract we provide a new formulation.

Abstract Voronoi diagrams (AVDs). AVDs were introduced by Klein [7]. Instead of sites and distance measures, they are defined in terms of bisecting curves that satisfy some

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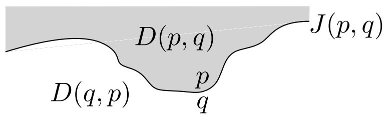

Figure 1 A bisector $J(p,q)$ and its dominance regions; $D(p,q)$ is shown shaded.

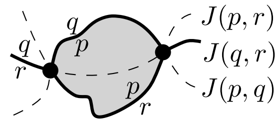

Figure 2 The Voronoi diagram $\mathcal{V}(\{p,q,r\})$ in solid lines. The shaded region is $\mathrm{VR}(p,\{p,q,r\})$.

simple combinatorial properties. Given a set S of n abstract sites, the bisector $J(p,q)$ of two sites $p,q \in S$ is an unbounded curve, homeomorphic to a line, that divides the plane into two open domains: the *dominance region of* p, $D(p,q)$ (with *label* p), and the *dominance region of* q, $D(q,p)$ (with *label* q), see Fig. 1. The Voronoi region of p is: $\mathrm{VR}(p,S) = \bigcap_{q\in S\setminus\{p\}} D(p,q)$ and the *(nearest-neighbor) abstract Voronoi diagram* of S is $\mathcal{V}(S) = \mathbb{R}^2 \setminus \bigcup_{p\in S} \mathrm{VR}(p,S)$, see Fig. 2. Following the traditional model of abstract Voronoi diagrams [7] an *admissible* system of bisectors $\mathcal{J}$ is assumed to satisfy the following axioms, for every subset $S' \subseteq S$:

(A1) Each nearest Voronoi region $\mathrm{VR}(p,S')$ is non-empty and pathwise connected.
(A2) Each point in the plane belongs to the closure of a nearest Voronoi region $\mathrm{VR}(p,S')$.
(A3) After stereographic projection to the sphere, each bisector is a Jordan curve through the north pole.
(A4) Any two bisectors $J(p,q)$ and $J(r,t)$ intersect transversally and in a finite number of points. (It is possible to relax this axiom, see [8]).

$\mathcal{V}(S)$ is a plane graph of structural complexity $O(n)$ and its regions are simply-connected. It can be computed in time $O(n\log n)$, randomized or deterministic, see [2]. To update $\mathcal{V}(S)$, after deleting one site $s \in S$, we compute $\mathcal{V}(S\setminus\{s\})$ within $\mathrm{VR}(s,S)$. The sequence of sites along $\partial\mathrm{VR}(s,S)$ forms a Davenport-Schinzel sequence (DSS) of order 2 and this constitutes a major difference from the respective problem for points where no repetition can occur.

Our results. We give a simple randomized algorithm to compute $\mathcal{V}(S\setminus\{s\})$ within $\mathrm{VR}(s,S)$ in expected time linear on the complexity of $\partial\mathrm{VR}(s,S)$. The algorithm is simple, not more complicated than its counterpart for points [4], and this is achieved by computing simplified intermediate structures. These are *Voronoi-like* diagrams, having a structure similar to an abstract Voronoi diagram, without however being such diagrams. We prove that Voronoi-like diagrams are well-defined and robust under an *insertion operation*, thus, making possible a randomized incremental construction for $\mathcal{V}(S\setminus\{s\})\cap\mathrm{VR}(s)$ in expected linear time. Our approach can be adapted (in fact, simplified) to compute, in expected linear time, the farthest abstract Voronoi diagram after the sequence of its faces at infinity is known; the latter can be computed in time $O(n\log n)$. Our technique can be applied to concrete diagrams that may not strictly fall under the AVD model such as Voronoi diagrams of line segments that may intersect and of planar straight-line graphs (including simple and non-simple polygons). For intersecting line segments, $\partial\mathrm{VR}(s,S)$ is a Davenport-Schinzel sequence of order 4 [10].

2 Problem formulation

Let S be a set of n abstract *sites* that define an admissible system of bisectors $\mathcal{J} = \{J(p,q) : p \neq q \in S\}$. Bisectors that have a site p in common are called *p-related*. Two related bisectors $J(p,q)$ and $J(p,r)$ can intersect at most twice; bisector $J(q,r)$ also intersects with them at the same point(s) [7]. Since related bisectors in $\mathcal{J}$ intersect at most twice, the sequence of site occurrences along $\partial\mathrm{VR}(p,S)$, $p \in S$, forms a DSS of order 2 (by [11, Theorem 5.7]).

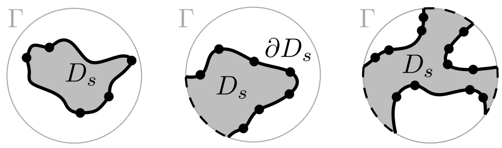

Figure 3 The domain $D_s = \mathrm{VR}(s, S) \cap D_\Gamma$.

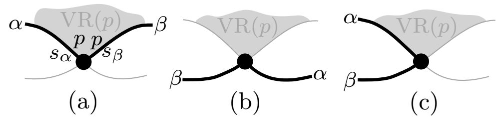

Figure 4 (a) Arcs α, β fulfill the p-monotone path condition; they do not fulfill it (b) and (c).

To update $\mathcal{V}(S)$ after deleting one site $s \in S$, we compute $\mathcal{V}(S \setminus \{s\})$ within $\mathrm{VR}(s, S)$, i.e., we compute $\mathcal{V}(S \setminus \{s\}) \cap \mathrm{VR}(s, S)$; its structure is given in the following lemma.

▶ **Lemma 1.** *$\mathcal{V}(S \setminus \{s\}) \cap VR(s, S)$ is a forest having exactly one face for each Voronoi edge of $\partial VR(s, S)$. Its leaves are the Voronoi vertices of $\partial VR(s, S)$, and points at infinity if $VR(s, S)$ is unbounded. If $VR(s, S)$ is bounded then $\mathcal{V}(S \setminus \{s\}) \cap VR(s, S)$ is a tree.*

We make a general position assumption that no three p-related bisectors intersect at the same point. This implies that Voronoi vertices have degree 3. We consider a closed Jordan curve Γ large enough to enclose all intersections of bisectors in $\mathcal{J}$, and such that each bisector crosses Γ exactly twice and transversally; the interior of Γ is denoted D_Γ. Our *domain* of computation is $D_s = \mathrm{VR}(s, S) \cap D_\Gamma$ and we compute $\mathcal{V}(S \setminus \{s\}) \cap D_s$, see Figure 3.

Let $\mathcal{S}$ denote the sequence of Voronoi edges along $\partial\mathrm{VR}(s, S)$, i.e., $\mathcal{S} = \partial\mathrm{VR}(s, S) \cap D_\Gamma$. Each arc $\alpha \in \mathcal{S}$ is induced by a site $s_\alpha \in S \setminus \{s\}$, where $\alpha \subseteq J(s, s_\alpha)$. We can interpret the arcs in $\mathcal{S}$ as sites that induce a Voronoi diagram $\mathcal{V}(\mathcal{S})$, where $\mathcal{V}(\mathcal{S}) = \mathcal{V}(S \setminus \{s\}) \cap D_s$, see Figure 7(a). In this respect, each arc $\alpha \in \mathcal{S}$ has a Voronoi region, $\mathrm{VR}(\alpha, \mathcal{S})$, which is the face of $\mathcal{V}(S \setminus \{s\}) \cap D_s$ incident to α.

In the remaining of this section we define a *Voronoi-like diagram* for a subset $\mathcal{S}'$ of arcs in $\mathcal{S}$ (Def. 2). To this aim we need some definitions. For a site $p \in S$ and $S' \subseteq S$, let $\mathcal{J}_{p,S'} = \{J(p, q) \mid q \in S', q \neq p\}$ denote the set of all *p-related* bisectors involving sites in S'.

A *path* P in a bisector system $\mathcal{J}_{p,S'}$ is a connected subset of alternating edges and vertices in the arrangement of $\mathcal{J}_{p,S'}$. An *arc* α of P is a maximal connected set, along P, of consecutive edges and vertices of the arrangement, which belong to the same bisector. The common endpoint of two consecutive arcs of P is a *vertex of* P; an arc of P is also called an *edge*. For an arc $\alpha \in P$, let $s_\alpha \in S$ be the site that *induces* α, i.e., $\alpha \subseteq J(p, s_\alpha)$.

A path P in $\mathcal{J}_{p,S'}$ is called *p-monotone*, if any two consecutive arcs $\alpha, \beta \in P$ correspond to the Voronoi edges in $\partial\mathrm{VR}(p, \{p, s_\alpha, s_\beta\})$ that are incident to the common endpoint of α, β (see Fig. 4). The *envelope of* $\mathcal{J}_{p,S'}$, with respect to site p, is the boundary of the Voronoi region $\mathrm{VR}(p, S' \cup \{p\})$, $\mathrm{env}(\mathcal{J}_{p,S'}) = \partial\mathrm{VR}(p, S' \cup \{p\})$. Fig. 5 illustrates two p-monotone paths, where (a) is an envelope. Notice, $\mathcal{S} = \mathrm{env}(\mathcal{J}_{s,S\setminus\{s\}}) \cap D_\Gamma$.

Consider $\mathcal{S}' \subseteq \mathcal{S}$ and let $S' = \{s_\alpha \in S \mid \alpha \in \mathcal{S}'\} \subseteq S \setminus \{s\}$ be its corresponding set of sites. A *boundary curve for* $\mathcal{S}'$ is a closed s-monotone path in $\mathcal{J}_{s,S'} \cup \Gamma$ that contains all arcs in $\mathcal{S}'$. Note that we include Γ in the definition of a boundary curve so that we unify the various connected components of $\mathcal{J}_{s,S'}$ and obtain a single curve. The part of the plane enclosed in

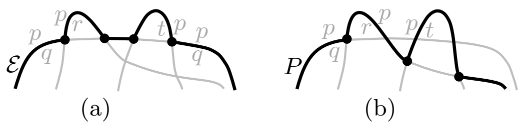

Figure 5 (a) The envelope $\mathcal{E} = \mathrm{env}(\mathcal{J}_{p,\{q,r,t\}})$. (b) A p-monotone path P in $\mathcal{J}_{p,\{q,r,t\}}$.

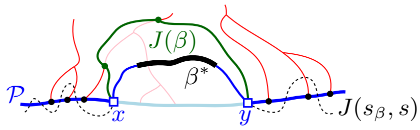

Figure 6 $\mathcal{P}_\beta = \mathcal{P} \oplus \beta$, core arc β^* is bold and black. The endpoints of $\beta \supseteq \beta^*$ are x and y.

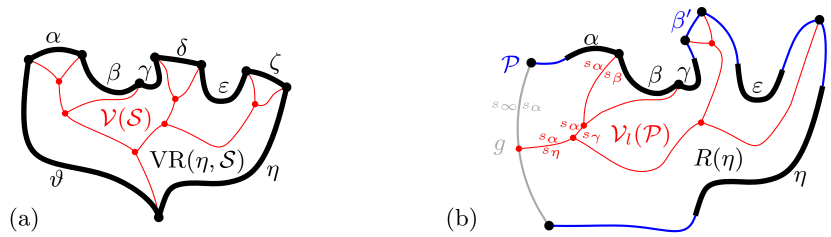

Figure 7 (a) illustrates $\mathcal{S}$ in black (bold) and $\mathcal{V}(\mathcal{S})$ in red, $\mathcal{S} = (\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \vartheta)$. (b) illustrates $\mathcal{V}_l(\mathcal{P})$ for boundary curve $\mathcal{P} = (\alpha, \beta, \gamma, \beta', \varepsilon, \eta, g)$. $\mathcal{S}' = (\alpha, \beta, \gamma, \varepsilon, \eta)$ is shown in bold. The arcs of $\mathcal{P}$ are original except the auxiliary arc β' and the Γ-arc g.

a boundary curve $\mathcal{P}$ is called the *domain* of $\mathcal{P}$, denoted by $D_{\mathcal{P}}$. Given $\mathcal{P}$, we also use $S_{\mathcal{P}}$ to denote the set of sites S'.

Figure 7(b) illustrates a boundary curve for $\mathcal{S}' \subset \mathcal{S}$, where $\mathcal{S}$ is shown bold in Figure 7(a). $\mathcal{S}'$ can admit several different boundary curves, one being the envelope $\text{env}(\mathcal{J}_{s,S'} \cup \Gamma)$. A boundary curve $\mathcal{P}$ consists of pieces of bisectors in $\mathcal{J}_{s,S'}$, called *boundary arcs*, and pieces of Γ, called Γ-*arcs*; Γ-arcs indicate the openings of the domain to infinity. Among the boundary arcs in $\mathcal{P}$, those that contain an arc of $\mathcal{S}'$ are called *original* and others are called *auxiliary arcs*. Original arcs are expanded versions of the arcs in $\mathcal{S}'$; to differenciate among them the arcs in $\mathcal{S}$ are called *core arcs* (shown bold in Figure 7).

▶ **Definition 2.** Given a boundary curve $\mathcal{P}$ in $\mathcal{J}_{s,S'} \cup \Gamma$, a *Voronoi-like diagram* of $\mathcal{P}$ is a plane graph on $\mathcal{J}(S') = \{J(p,q) \in \mathcal{J} \mid p, q \in S'\}$ inducing a subdivision on the domain $D_{\mathcal{P}}$ as follows (see Figure 7(b)): (1) There is exactly one face $R(\alpha)$ for each boundary arc α of $\mathcal{P}$; $\partial R(\alpha)$ consists of the arc α and an s_α-monotone path in $\mathcal{J}_{s_\alpha,S'} \cup \Gamma$. (2) $\bigcup_{\alpha \in \mathcal{P} \setminus \Gamma} \overline{R(\alpha)} = \overline{D_{\mathcal{P}}}$. The Voronoi-like diagram of $\mathcal{P}$ is $\mathcal{V}_l(\mathcal{P}) = D_{\mathcal{P}} \setminus \bigcup_{\alpha \in \mathcal{P}} R(\alpha)$.

In the full paper, we prove that Voronoi-like regions are related to real Voronoi regions as supersets. For example, in Figure 7, the Voronoi-like region $R(\eta)$ (shown in 7(b)) is a superset of Voronoi region $\text{VR}(\eta, \mathcal{S})$ in 7(a); similarly for $R(\alpha)$. Real Voronoi regions are induced by the envelope $\mathcal{E}$ of $\mathcal{S}'$, where $\mathcal{E} = \text{env}(\mathcal{J}_{s,S'} \cup \Gamma)$, and $\mathcal{V}(\mathcal{E}) = \mathcal{V}(S') \cap D_{\mathcal{E}}$. It is not hard to see that $\mathcal{V}(\mathcal{E}) = \mathcal{V}_l(\mathcal{E})$. Thus, $\mathcal{V}_l(\mathcal{S})$ coincides with the real Voronoi diagram $\mathcal{V}(\mathcal{S})$.

In the full paper, we also prove that the Voronoi-like diagram of a boundary curve is unique (if it exists). The complexity of $\mathcal{V}_l(\mathcal{P})$ is $O(|\mathcal{P}|)$, where $|\cdot|$ denotes complexity, since it is a planar graph with exactly one face per boundary arc and vertices of degree 3 (or 1).

3 Insertion in a Voronoi-like diagram

Consider a boundary curve $\mathcal{P}$ for $\mathcal{S}' \subset \mathcal{S}$ and its Voronoi-like diagram $\mathcal{V}_l(\mathcal{P})$ within the domain $D_{\mathcal{P}}$. Let β^* be an arc in $\mathcal{S} \setminus \mathcal{S}'$, thus, β^* is contained in the closure of the domain $\overline{D_{\mathcal{P}}}$. We define arc $\beta \supseteq \beta^*$ as the connected component of $J(s, s_\beta) \cap \overline{D_{\mathcal{P}}}$ that contains β^* (see Figure 6). We also define an insertion operation $\oplus$ that inserts arc β in $\mathcal{P}$, deriving a new boundary curve $\mathcal{P}_\beta = \mathcal{P} \oplus \beta$, and inserts $R(\beta)$ in $\mathcal{V}_l(\mathcal{P})$, deriving $\mathcal{V}_l(\mathcal{P}_\beta) = \mathcal{V}_l(\mathcal{P}) \oplus \beta$. $\mathcal{P}_\beta$ is the boundary curve obtained by deleting the portion of $\mathcal{P} \cap D(s_\beta, s)$ between the endpoints of β and substituting it with β, see Figure 6. Figure 8 enumerates the possible cases of $\mathcal{P} \oplus \beta$ which is summarized in the following observation.

▶ Observation 3. Possible cases of inserting arc β in $\mathcal{P}$, see Figure 8. $D_{\mathcal{P}_\beta} \subseteq D_{\mathcal{P}}$.

(a) β straddles the endpoint of two consecutive boundary arcs; no arcs in $\mathcal{P}$ are deleted.

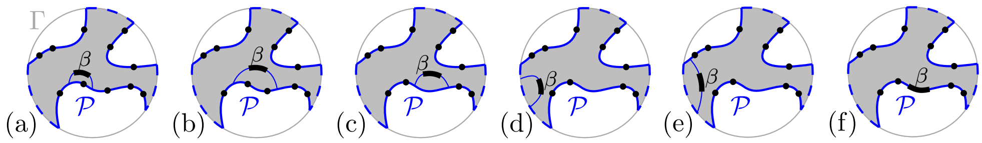

Figure 8 Insertion cases for an arc $\beta \supseteq \beta^*$. The domain $D_\mathcal{P}$ is shown shaded.

(b) (Auxiliary) arcs in $\mathcal{P}$ are deleted by β; their regions are also deleted from $\mathcal{V}_l(\mathcal{P}_\beta)$.
(c) An arc $\alpha \in \mathcal{P}$ is split into two arcs by β; $R(\alpha)$ in $\mathcal{V}_l(\mathcal{P})$ will also be split.
(d) A Γ-arc is split in two; $\mathcal{V}_l(\mathcal{P}_\beta)$ may switch from being a tree to being a forest.
(e) A Γ-arc is deleted or shrunk by inserting β. $\mathcal{V}_l(\mathcal{P}_\beta)$ may become a tree.
(f) $\mathcal{P}$ already contains a boundary arc $\bar\beta \supseteq \beta^*$; then $\beta = \bar\beta$ and $\mathcal{P}_\beta = \mathcal{P}$.

Note that $\mathcal{P}_\beta$ may contain fewer, the same number, or even one extra auxiliary arc compared to $\mathcal{P}$.

▶ **Lemma 4.** *The curve $\mathcal{P}_\beta = \mathcal{P} \oplus \beta$ is a boundary curve for $\mathcal{S}' \cup \{\beta^*\}$.*

Given $\mathcal{V}_l(\mathcal{P})$ and arc β, where $\beta^* \in \mathcal{S} \setminus \mathcal{S}'$, we define a *merge curve* $J(\beta)$, within $D_\mathcal{P} \cup \Gamma$, which delimits $\partial R(\beta)$ in $\mathcal{V}_l(\mathcal{P}_\beta)$, see Figure 6. We define $J(\beta)$ incrementally, starting at an endpoint of β. Let x and y denote the endpoints of β, where x, β, y are assumed in counterclockwise order around $\mathcal{P}_\beta$.

▶ **Definition 5.** Given $\mathcal{V}_l(\mathcal{P})$ and arc $\beta \subset J(s, s_\beta)$, the *merge curve* $J(\beta)$ is a path $(v_1, \dots, v_m)$ in the arrangement of s_β-related bisectors $\mathcal{J}_{s_\beta, S_\mathcal{P}} \cup \Gamma$, connecting the endpoints of β, $v_1 = x$ and $v_m = y$. Each edge $e_i = (v_i, v_{i+1})$ is an arc of a bisector $J(s_\beta, \cdot)$ or an arc on Γ. For $i = 1$: if $x \in J(s_\beta, s_\alpha)$, then $e_1 \subseteq J(s_\beta, s_\alpha)$; if $x \in \Gamma$, then $e_1 \subseteq \Gamma$. Given v_i, vertex v_{i+1} and edge e_{i+1} are defined as follows. (Wlog we assume a clockwise ordering of $J(\beta)$).

1. If $e_i \subseteq J(s_\beta, s_\alpha)$, let v_{i+1} be the other endpoint of the component $J(s_\beta, s_\alpha) \cap R(\alpha)$ incident to v_i. If $v_{i+1} \in J(s_\beta, \cdot) \cap J(s_\beta, s_\alpha)$, then $e_{i+1} \subseteq J(s_\beta, \cdot)$. If $v_{i+1} \in \Gamma$, then $e_{i+1} \subseteq \Gamma$.
2. If $e_i \subseteq \Gamma$, let g be the Γ-arc incident to v_i. Let $e_{i+1} \subseteq J(s_\beta, s_\gamma)$, where $R(\gamma)$ is the first region, incident to g clockwise from v_i, such that $J(s_\beta, s_\gamma)$ intersects $g \cap \overline{R(\gamma)}$; let v_{i+1} be this intersection point.

In the full paper we prove the following theorem, which shows that $J(\beta)$ is well defined.

▶ **Theorem 6.** *$J(\beta)$ is a unique s_β-monotone path in $\mathcal{J}_{s_\beta, S_\mathcal{P}} \cup \Gamma$, which connects the endpoints of β. $J(\beta)$ can contain at most one edge per region of $\mathcal{V}_l(\mathcal{P})$, with the exception of the first and last edge, if v_1 and v_m are incident to the same face in $\mathcal{V}_l(\mathcal{P})$. $J(\beta)$ cannot intersect the interior of arc β.*

We define $R(\beta)$ as the area enclosed by $\beta \cup J(\beta)$. Let $\mathcal{V}_l(\mathcal{P}) \oplus \beta = \big((\mathcal{V}_l(\mathcal{P}) \setminus R(\beta)) \cup J(\beta)\big) \cap D_{\mathcal{P}_\beta}$ be the subdivision of $D_{\mathcal{P}_\beta}$ obtained by inserting $J(\beta)$ in $\mathcal{V}_l(\mathcal{P})$ and deleting any portion of $\mathcal{V}_l(\mathcal{P})$ enclosed by $J(\beta)$.

▶ **Theorem 7.** *$\mathcal{V}_l(\mathcal{P}) \oplus \beta$ is a Voronoi-like diagram for $\mathcal{P}_\beta = \mathcal{P} \oplus \beta$, denoted $\mathcal{V}_l(\mathcal{P}_\beta)$.*

The time complexity to compute $J(\beta)$ and update $\mathcal{V}_l(\mathcal{P}_\beta)$ is as follows: Let $\tilde{\mathcal{P}}$ denote the finer version of $\mathcal{P}$ as obtained by intersecting $\mathcal{P}$ with $\mathcal{V}_l(\mathcal{P})$. $|\tilde{\mathcal{P}}|$ is $O(|\mathcal{P}|)$, since $|\mathcal{V}_l(\mathcal{P})|$ is $O(|\mathcal{P}|)$. Let α and γ be the first original arcs on $\mathcal{P}_\beta$ occurring before and after β. Let $d(\beta)$ be the number of arcs in $\tilde{\mathcal{P}}$ between α and γ (both boundary and Γ-arcs). Given α, γ, and $\mathcal{V}_l(\mathcal{P})$, in all cases of Observation 3, except (c), the merge curve $J(\beta)$ and the diagram $\mathcal{V}_l(\mathcal{P}_\beta)$ can be computed in time $O(|R(\beta)| + d(\beta))$. In case (c), where an arc is split and a new arc ω is created by the insertion of β, the time is $O(|\partial R(\beta)| + |\partial R(\omega)| + d(\beta))$.

4 A randomized incremental algorithm

Consider a random permutation of the set of arcs $\mathcal{S}$, $o = (\alpha_1, \ldots, \alpha_h)$. For $1 \leq i \leq h$ define $\mathcal{S}_i = \{\alpha_1, \ldots, \alpha_i\} \subseteq \mathcal{S}$ to be the subset of the first i arcs in o. Given $\mathcal{S}_i$, let $\mathcal{P}_i$ denote a boundary curve for $\mathcal{S}_i$, which induces a domain $D_i = D_{\mathcal{P}_i}$. The randomized algorithm is inspired by the randomized, two-phase, approach of Chew [4] for the Voronoi diagram of points in convex position; however, it constructs Voronoi-like diagrams of boundary curves $\mathcal{P}_i$ within a series of shrinking domains $D_i \supseteq D_{i+1}$. In phase 1, the arcs in $\mathcal{S}$ get deleted one by one in reverse order of o, while recording the neighbors of each deleted arc at the time of its deletion. Let $\mathcal{P}_1 = \partial(D(s, s_{\alpha_1}) \cap D_\Gamma)$ and $D_1 = D(s, s_{\alpha_1}) \cap D_\Gamma$. Let $R(\alpha_1) = D_1$. $\mathcal{V}_l(\mathcal{P}_1) = \emptyset$ is the Voronoi-like diagram for $\mathcal{P}_1$. In phase 2, we start with $\mathcal{V}_l(\mathcal{P}_1)$ and incrementally compute $\mathcal{V}_l(\mathcal{P}_{i+1})$, $i = 1, \ldots, h-1$, by inserting arc α_{i+1} in $\mathcal{V}_l(\mathcal{P}_i)$, where $\mathcal{P}_{i+1} = \mathcal{P}_i \oplus \alpha_{i+1}$ and $\mathcal{V}_l(\mathcal{P}_{i+1}) = \mathcal{V}_l(\mathcal{P}_i) \oplus \alpha_{i+1}$. At the end, we obtain $\mathcal{V}_l(\mathcal{P}_h)$, where $\mathcal{P}_h = \mathcal{S}$. We have already established that $\mathcal{V}_l(\mathcal{S}) = \mathcal{V}(\mathcal{S})$, thus, the algorithm is correct. $\mathcal{P}_i$ may contain at most $2i$ arcs (see Observation 3), thus, the complexity of $\mathcal{V}_l(\mathcal{P}_i)$ is $O(i)$.

Given the results on Voronoi-like diagrams in Sections 2 and 3, the time analysis becomes similar to the one for the farthest-segment Voronoi diagram [6], with some additional cases to consider since $\mathcal{V}_l(\mathcal{P}_i)$ is a forest and not necessarily a tree.

▶ **Lemma 8.** *The expected number of arcs in $\tilde{\mathcal{P}}_i$ (auxiliary boundary arcs and fine Γ-arcs) that are visited while inserting α_{i+1} is $O(1)$.*

▶ **Theorem 9.** *Given an abstract Voronoi diagram $\mathcal{V}(S)$, $\mathcal{V}(S \setminus \{s\}) \cap VR(s, S)$ can be computed in expected $O(h)$ time, where h is the complexity of $\partial VR(s, S)$. Thus, $\mathcal{V}(S \setminus \{s\})$ can also be computed in expected time $O(h)$.*

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