Integer and Mixed Integer Tverberg Numbers

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Abstract

We show the exact values of Tverberg numbers of $\mathbb{Z}^2$ and improve the bounds for $\mathbb{Z}^3$ and $\mathbb{Z}^j \times \mathbb{R}^k$.

1 Introduction

Consider n points in $\mathbb{R}^d$ and a positive integer $m \geq 2$. If $n \geq (m-1)(d+1)+1$, the points can always be partitioned into m subsets whose convex hulls contain a common point. This is the celebrated theorem of Tverberg [11], which has been the topic of many generalizations and variations since it was first proved in 1966. In this paper we formalize new versions of Tverberg's theorem where the coordinates of the points are integer. Our opening result closes a gap in the literature. It deals with a Tverberg-type theorem in the case of $\mathbb{Z}^2$. According to Eckhoff [6] it was stated by Doignon in a conference. Doignon (personal communication) confirmed that this was not published.

▶ **Theorem 1.** *Consider n points in $\mathbb{Z}^2$ and a positive integer $m \geq 3$. If $n \geq 4m-3$, then the points can be partitioned into m subsets whose convex hulls contain a common point in $\mathbb{Z}^2$.*

Such a partition is an *integer m-Tverberg partition* and such a common point is an *integer Tverberg point* for that partition. Regarding the case $m = 2$, the integer 2-Tverberg partitions are *integer Radon partitions*. Any configuration of at least 6 points admits an integer Radon partition. This was proved by Doignon in his PhD thesis [5] and later discovered independently by Onn [10]. All these values are optimal as shown by following examples. The 5-point configuration $\{(0,0),(0,1),(2,0)(1,2),(3,2)\}$, exhibited by Onn in the cited paper, has no Radon partition. To address the optimality when $m \geq 3$, consider the set $\{(i,i),(i,-i+1)\colon i = -m+2,-m+3,\ldots,m-2,m-1\}$. (According to Eckhoff [6], this set was proposed by Doignon during a conference.) It has $4m-4$ points and a moment of reflection might convince the reader that it has no integer m-Tverberg partition.

More generally, one can define the *Tverberg number* $\mathrm{Tv}(S,m)$ for a subset S of $\mathbb{R}^d$ and an integer $m \geq 2$ as the smallest integer number n such that any multiset of n points in S admits a partition into m subsets $A_1, A_2, \ldots, A_m$ with

$$\left(\bigcap_{i=1}^{m} \mathrm{conv}(A_i)\right) \cap S \neq \varnothing.$$

(Here, by "partition of a multiset", we mean that each element of a multiset A is contained in a number of subsets that does not exceed its multiplicity in A.) Theorem 1 together with the discussion that follows can then be rephrased as

$$\mathrm{Tv}(\mathbb{Z}^2, m) = \begin{cases} 6 & \text{if } m = 2, \\ 4m-3 & \text{otherwise.} \end{cases}$$

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This is an extended abstract of a presentation given at EuroCG'18. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

Our second main result improves the upper bound for the case $S = \mathbb{Z}^3$.

▶ **Theorem 2.** *The following inequality holds for all $m \geq 2$:*

$$\mathrm{Tv}(\mathbb{Z}^3, m) \leq 24m - 31.$$

Proofs of Theorems 1 and 2 are respectively given in Sections 2 and 3. The strategy of both proofs is standard: we show that there exists an integer centerpoint (which we define at the end of this section) of sufficient depth and that this centerpoint is actually a *Tverberg point* of an m-Tverberg partition.

Choosing S of the form $\mathbb{Z}^j \times \mathbb{R}^k$ leads to the "mixed integer" case, which is the common generalization of the real and the integer cases. Our third main result is an inequality simultaneously involving the three already considered instantiations of S: real, integer, and mixed integer.

▶ **Theorem 3.** *The following inequality holds for all positive integers j and k and all $m \geq 2$:*

$$\mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) \leq \mathrm{Tv}(\mathbb{Z}^j, \mathrm{Tv}(\mathbb{R}^k, m)).$$

Finally, in Section 4, we prove Theorem 3 and collect some consequences of the main theorems presented above, including the following result:

$$2^j(m-1)(k+1) + 1 \leq \mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) \leq j2^j(m-1)(k+1) + 1. \tag{1}$$

To conclude the introduction we mention a key lemma about integral centerpoints that is used for proving Theorems 1 and 2. Given a multiset A of points, a point $\boldsymbol{p}$ is a *centerpoint of depth σ* in A if every closed half-space containing $\boldsymbol{p}$ contains at least σ points of A.

▶ **Lemma 4.** *Consider a multiset A of points in $\mathbb{Z}^d$. If $|A| \geq 2^d(m-1)+1$ (counting multiplicities), then there is a centerpoint $\boldsymbol{p} \in \mathbb{Z}^d$ of depth m in A.*

Although the present version is new, similar lemmas have been used throughout the literature and their proofs typically rely on some version of Helly's theorem [7]. We omit the classical details here, and simply mention that we need the following theorem of Doignon [4]: If $\mathcal{F}$ is a finite family of at least 2^d convex subsets of $\mathbb{Z}^d$ such that any 2^d members of $\mathcal{F}$ have an intersection point in $\mathbb{Z}^d$, there is a point $\boldsymbol{p} \in \mathbb{Z}^d$ in every set in $\mathcal{F}$.

In Sections 2 and 3 when we refer to Tverberg partitions or Tverberg points we focus on *integer* Tverberg partitions.

Related Results from the Literature

The problem of computing the Tverbeg number for $\mathbb{Z}^d$ with $d \geq 3$ seems to be challenging. It has been identified as an interesting problem since the 1970's and yet the following inequalities are almost all that is known about this problem: For the general case, De Loera et al. [8] proved

$$2^d(m-1) + 1 \leq \mathrm{Tv}(\mathbb{Z}^d, m) \leq d2^d(m-1) + 1 \quad \text{for } d \geq 1 \text{ and } m \geq 2. \tag{2}$$

Two special cases get better bounds:

$$\mathrm{Tv}(\mathbb{Z}^3, 2) \leq 17 \qquad \text{and} \qquad 5 \cdot 2^{d-2} + 1 \leq \mathrm{Tv}(\mathbb{Z}^d, 2) \quad \text{for } d \geq 2. \tag{3}$$

The left-hand side inequality is due to Bezdek and Blokhuis [2] and the right-hand side was proved by Doignon in his PhD thesis (and rediscovered by Onn).

The bounds for the "mixed integer" case include the bounds for the Radon number (2-Tverberg number) found by Averkov and Weismantel [1].

$$2^j(k+1)+1 \le \mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, 2) \le (j+k)2^j(k+1) - j - k + 2.$$

Later, De Loera et al. [8] gave the following general bound for all Tverberg numbers:

$$\mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) \le (j+k)2^j(m-1)(k+1)+1.$$

Note that (1) above is a simultaneous improvement of both of these.

2 Tverberg Numbers over $\mathbb{Z}^2$: Proof of Theorem 1

The theorem will follow easily from the following two lemmas, the first covering the case $m \ge 3$ and the second the case $m = 2$.

▶ **Lemma 5.** *Consider a multiset A of points in $\mathbb{Z}^2$ with $|A| \ge 4m-3$ and $m \ge 3$. If $\boldsymbol{p} \notin A$ is a centerpoint of depth m, then there is an m-Tverberg partition with $\boldsymbol{p}$ as Tverberg point.*

▶ **Lemma 6.** *Consider a multiset A of points in $\mathbb{Z}^2$ with $|A| \ge 6$. If $\boldsymbol{p} \notin A$ is a centerpoint of depth two, then there is a Radon partition with $\boldsymbol{p}$ as Tverberg point.*

Proof of Theorem 1. Consider a multiset A of at least $4m-3$ points in $\mathbb{Z}^2$. By Lemma 4, A has an integer centerpoint $\boldsymbol{p}$ of depth m. If $\boldsymbol{p}$ is an element of A with multiplicity $\mu \ge 0$, then take the singletons $\{\boldsymbol{p}\}$ as μ of the sets in the Tverberg partition. Then $\boldsymbol{p}$ is a centerpoint of depth $m-\mu$ of the remaining $4m-\mu-3$ points. If $\mu \ge m$, we are done, and if $\mu = m-1$, the point $\boldsymbol{p}$ is in the convex hull of the remaining points and we take them to be the last set in the desired partition. If $\mu \le m-3$, according to Lemma 5, there is an $(m-\mu)$-Tverberg partition of the remaining points with $\boldsymbol{p}$ as Tverberg point. There is thus an m-Tverberg partition of A with $\boldsymbol{p}$ as Tverberg point. The case $\mu = m-2$ is treated similarly with the help of Lemma 6 in place of Lemma 5 ◀

Proof of Lemma 5. Since $\boldsymbol{p}$ is not in A, up to a radial projection, we can assume that the points of A are arranged in a circle around $\boldsymbol{p}$. Define q and r to be respectively the quotient and the remainder of the Euclidean division of $|A|$ by m. Define moreover e to be $\lceil \frac{r}{q} \rceil$.

Suppose first that $\boldsymbol{p}$ is a centerpoint of depth $m+e$. In such a case, we arbitrarily select a first point in A, and label clockwise the points with elements in $[m]$ according to the following pattern:

$$1,2,\ldots,m,1,2,\ldots,e,1,2,\ldots,m,1,2,\ldots,e,1,2,\ldots m,1,2,\ldots k,$$

where $k = |A| - qm - (q-1)e$. Note that we have $k \le e$. Each half-plane delimited by a line passing through $\boldsymbol{p}$ contains at least $m+e$ consecutive points in this pattern and thus has at least one point with each of the m different labels. Partitioning the points so that each subset consists of all points with a fixed label, we therefore obtain an m-Tverberg partition with $\boldsymbol{p}$ as Tverberg point.

Suppose now that $\boldsymbol{p}$ is not a centerpoint of depth $m+e$. There is thus a closed half-plane H_+ delimited by a line passing through $\boldsymbol{p}$ with $|H_+ \cap A| < m+e$. The complementary closed half-plane to H_+, which we denote by H_-, is such that $|H_- \cap A| > 4m-3-(m+e)$. Define ℓ to be $|H_- \cap A|$. Since $e \le \frac{m}{3}$, we have $\ell \ge 2m$. Denote the points in $H_- \cap A$ by $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_\ell$, where the indices are increasing when we move clockwise. We label $\boldsymbol{x}_i$ with $r+i$ from $\boldsymbol{x}_1$ to $\boldsymbol{x}_{m-r}$, and then label $\boldsymbol{x}_{m-r+j}$ with j from $\boldsymbol{x}_{m-r+1}$ to $\boldsymbol{x}_m$. We then continue labeling

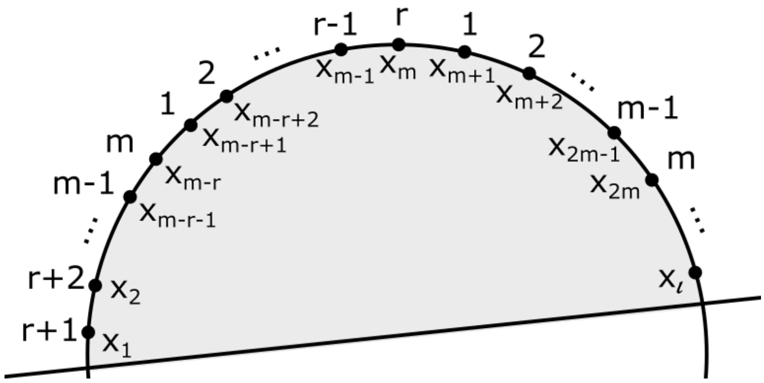

Figure 1 Labeling of the points in the half-plane H_-.

the points of A, still moving clockwise, using labels $1, 2, \ldots, m, \ldots, 1, 2, \ldots m, 1, 2, \ldots r$. See Figure 1 for an illustration of the labeling scheme.

The labeling pattern is such that any sequence of m consecutive points either has all m labels, or contains the two consecutive points $\boldsymbol{x}_m$ and $\boldsymbol{x}_{m+1}$. Let us prove that any closed half-plane H delimited by a line passing through $\boldsymbol{p}$ contains at least one point with each label. Once this is proved, the conclusion will be immediate by taking as subsets of points those with same labels, as above.

If such an H does not simultaneously contain $\boldsymbol{x}_m$ and $\boldsymbol{x}_{m+1}$, then H contains at least one point with each label. Consider thus a closed half-plane H delimited by a line passing through $\boldsymbol{p}$ and containing $\boldsymbol{x}_m$ and $\boldsymbol{x}_{m+1}$. Note that according to Farkas' lemma, $\boldsymbol{x}_{m+1}$ cannot be separated from $\boldsymbol{x}_1$ and $\boldsymbol{x}_\ell$ by a line passing through $\boldsymbol{p}$, since they are all in H_-. This means that either H contains $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{m+1}$, or H contains $\boldsymbol{x}_{m+1}, \boldsymbol{x}_{m+2}, \ldots, \boldsymbol{x}_\ell$. In any case, H contains a point with each label. ◀

The proof of Lemma 6 is similar and left to the reader for brevity.

3 Tverberg Numbers over $\mathbb{Z}^3$: Proof of Theorem 2

We will make use of the following two lemmas. Lemma 7 is a consequence, upon close inspection of the argument, of the proof of the main theorem in the already mentioned paper by Bezdek and Blokhuis [2].

▶ **Lemma 7.** *Consider a multiset A of at least 17 points in $\mathbb{R}^3$ and a centerpoint $\boldsymbol{p}$ of depth 3 in A. There is a bipartition of A into two subsets whose convex hulls contain $\boldsymbol{p}$.*

▶ **Lemma 8.** *Consider a multiset A of points in $\mathbb{R}^3$ with $|A| \geq 24m - 31$ and $m \geq 2$. If $\boldsymbol{p} \notin A$ is a centerpoint of depth $3m-3$, then there is an m-Tverberg partition of A with $\boldsymbol{p}$ as Tverberg point.*

Proof. Since $\boldsymbol{p}$ is not an element of A, we assume without loss of generality that the points of A are located on a sphere centered at $\boldsymbol{p}$, as in the proof of Lemma 5.

We claim that we can find pairwise disjoint subsets $X_1, X_2, \ldots, X_{m-2}$ of A, each having $\boldsymbol{p}$ in its convex hull and each being of cardinality at most four. (Here "pairwise disjoint" means that each element of A is present in a number of X_i's that does not exceed its multiplicity in A.) We proceed by contradiction. Suppose that we can find at most $s < m-2$ such subsets X_i's. Then, by Carathéodory's theorem [3], $\boldsymbol{p}$ is not in the convex hull of the remaining points in A.Therefore there is a half-space H_+ delimited by a plane containing $\boldsymbol{p}$ such that $H_+ \cap A \subseteq \bigcup_{i=1}^s X_i$. On the other hand, since each X_i contains $\boldsymbol{p}$ in its convex hull (and we can assume the X_i are minimal with respect to containing $\boldsymbol{p}$), we have $|H_+ \cap X_i| \leq 3$ for all $i \in [s]$. Therefore $|H_+ \cap A| \leq |H_+ \cap (\bigcup_{i=1}^s X_i)| \leq 3s < 3(m-2)$, which is a contradiction since $\boldsymbol{p}$ is a centerpoint of depth $3m-3$ in A. There are thus $m-2$ disjoint subsets $X_1, X_2, \ldots, X_{m-2}$ as claimed.

Let X denote $\bigcup_{i=1}^{m-2} X_i$. Consider an arbitrary half-space H_+ delimited by a plane containing $\boldsymbol{p}$. Since $|H_+ \cap X_i| \leq 3$ for all i, we have $|H_+ \cap X| \leq 3(m-2)$. Furthermore $|H_+ \cap A| \geq 3m-3$, so $|H_+ \cap (A \setminus X)| \geq 3$. Since H_+ is arbitrary, $\boldsymbol{p}$ is a centerpoint of depth 3 of $A \setminus X$. Also, $|A \setminus X| \geq |A| - 4(m-2) \geq 20m - 23 \geq 17$, so Lemma 7 implies that $A \setminus X$ can be partitioned into two sets whose convex hulls contain $\boldsymbol{p}$. With the subsets X_i, we have therefore an m-Tverberg partition of A, with $\boldsymbol{p}$ as Tverberg point. ◀

From these two lemmas we can now finish the proof of Theorem 2.

Proof of Theorem 2. Consider a multiset A of $24m-31$ points in $\mathbb{Z}^3$. The case $m = 2$ is the already mentioned result by Bezdek and Blokhuis. Assume that $m \geq 3$. Applying Lemma 4, A has an integer centerpoint $\boldsymbol{p}$ of depth $3m-3$. If $\boldsymbol{p}$ is an element of A with multiplicity $\mu \geq 0$, then take the singletons $\{\boldsymbol{p}\}$ as μ of the sets in the Tverberg partition.

If $\mu \geq m$, we are done. If $\mu = m-1$, the point $\boldsymbol{p}$ is still in the convex hull of points in A, and thus we are done. And if $\mu \leq m-2$, the point $\boldsymbol{p}$ is still a centerpoint of depth $3m - \mu - 3 \geq 3(m-\mu) - 3$ of the remaining $24m - \mu - 31 \geq 24(m-\mu) - 31$ points. Thus, we may apply Lemma 8 to get an $(m-\mu)$-Tverberg partition of the remaining points, with $\boldsymbol{p}$ as Tverberg point, and conclude the result. ◀

4 Tverberg Numbers over $\mathbb{Z}^j \times \mathbb{R}^k$

In this section, we prove Theorem 3. We adapt an approach by Mulzer and Werner [9, Lemma 2.3] and show how all the results of our paper can be combined to improve known bounds and to determine new exact values for the Tverberg number in the mixed integer case.

Proof of Theorem 3. Let $t = \mathrm{Tv}(\mathbb{R}^k, m) = (m-1)(k+1)+1$. Choose a multiset A in $\mathbb{Z}^j \times \mathbb{R}^k$ with $|A| \geq \mathrm{Tv}(\mathbb{Z}^j, t)$. It suffices to prove that A can be partitioned into m subsets whose convex hulls contain a common point in $\mathbb{Z}^j \times \mathbb{R}^k$.

Let A' be the projection of A onto $\mathbb{Z}^j$. Since $|A'| \geq \mathrm{Tv}(\mathbb{Z}^j, t)$, there is a partition of A' into t subsets $Q'_1, \ldots, Q'_t$ whose convex hulls contain a common point $\boldsymbol{q}$ in $\mathbb{Z}^j$. The Q'_i are the projections onto $\mathbb{Z}^j$ of t disjoint subsets Q_i forming a partition of A. For each $i \in [t]$, we can find a point $\boldsymbol{q}_i \in \mathrm{conv}(Q_i)$ projecting onto $\boldsymbol{q}$.

The t points $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_t$ belong to $\{\boldsymbol{q}\} \times \mathbb{R}^k$. As $t = \mathrm{Tv}(\mathbb{R}^k, m)$, there exists a partition of $[t]$ into $I_1, \ldots, I_m$ and a point $\boldsymbol{p} \in \{\boldsymbol{q}\} \times \mathbb{R}^k$ such that $\boldsymbol{p} \in \mathrm{conv}\left(\bigcup_{i \in I_\ell} \boldsymbol{q}_i\right)$ for all $\ell \in [m]$. For each $\ell \in [m]$, define A_ℓ to be $\bigcup_{i \in I_\ell} Q_i$. We have for each $\ell \in [m]$

$$\boldsymbol{p} \in \mathrm{conv}\left(\bigcup_{i \in I_\ell} \boldsymbol{q}_i\right) \subseteq \mathrm{conv}\left(\bigcup_{i \in I_\ell} \mathrm{conv}(Q_i)\right) = \mathrm{conv}(A_\ell)$$

and the A_ℓ form the desired partition. ◀

Here are the new bounds and exact values we get:

(a) $\mathrm{Tv}(\mathbb{Z} \times \mathbb{R}^k, m) = 2(m-1)(k+1)+1$.

(b) $\mathrm{Tv}(\mathbb{Z}^2 \times \mathbb{R}^k, m) = 4(m-1)(k+1)+1$.
(c) $\mathrm{Tv}(\mathbb{Z}^3 \times \mathbb{R}^k, m) \leq 24(m-1)(k+1)-7$.
(d) $2^j(m-1)(k+1)+1 \leq \mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) \leq j2^j(m-1)(k+1)+1$.

The lower bound in (d) is obtained by repeated applications of Lemma 9 below, whose proof, almost identical to that of Proposition 2.1 in [10], is omitted for brevity. The upper bounds follow from Theorem 3, combined with the fact that $\mathrm{Tv}(\mathbb{Z}, m) = 2m-1$ (consider the median), Theorem 1, Theorem 2, and the upper bound in Equation (2), respectively.

▶ **Lemma 9.** *Let j and k be two non-negative integers. Then we have*

$$\mathrm{Tv}(\mathbb{Z}^{j+1} \times \mathbb{R}^k, m) > 2\,\mathrm{Tv}(\mathbb{Z}^j \times \mathbb{R}^k, m) - 2.$$

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