A Note on Planar Monohedral Tilings*

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Abstract

A planar *monohedral tiling* is a decomposition of $\mathbb{R}^2$ into congruent *tiles*. We say that such a tiling has the *flag property* if for each triple of tiles that intersect pairwise, the three tiles intersect in a common point. We show that for convex tiles, there exist only three classes of tilings that are not flag, and they all consist of triangular tiles; in particular, each convex tiling using polygons with $n \geq 4$ vertices is flag. We also show that an analogous statement for the case of non-convex tiles is not true by presenting a family of counterexamples.

1 Introduction

Problem statement and results. A *plane tiling* in the plane is a countable family of planar sets $\{T_1, T_2, \ldots\}$, called *tiles*, such that each T_i is compact and connected, the union of all T_i is the entire plane and the T_i are pairwise interior-disjoint. We call such a tiling *monohedral* if each T_i is congruent to T_1. In other words, a monohedral tiling can be obtained from the shape T_1 by repeatedly placing (translated, rotated, or reflected) copies of T_1. Two of the simplest examples for such monohedral tilings are shown in Figure 1. These are also instances of *convex tilings*, where we require that each tile is convex. A comprehensive study of tilings with numerous examples can be found in the monograph by Grünbaum and Shephard [3].

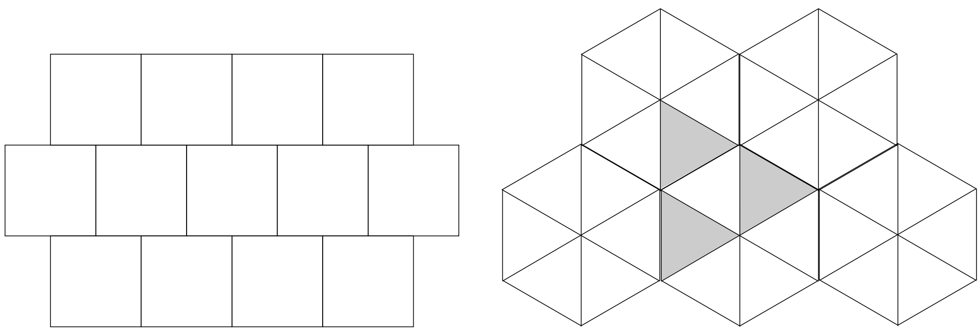

Figure 1 Monohedral tiling with squares (left) and equilateral triangles (right). On the right, an obstructing triple for the flag property is shaded.

We are interested in a special property of (monohedral) tilings: We say that a tiling is *flag* if whenever three tiles intersect pairwise, they also intersect in a point common to all three tiles. It can easily be verified that the left tiling in Figure 1 is flag, whereas the right

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tiling is not: the three edge neighbors of any triangle intersect pairwise (in single points), but have no common intersection. We call such a triple an *obstructing triple*. We are interested in the following question: which monohedral tilings have the flag property?

Our main result is that "most" convex monohedral tilings in the plane are flag. There are only three types of counterexamples, namely the ones depicted in Figure 1 (right) and in Figure 2. In particular, all counterexamples require triangles as tiles. As a consequence, every convex monohedral tiling with convex polygons having 4 or more vertices is flag.

To explain the three types of non-flag tilings, we observe that the union of the three tiles of an obstructing triple divides the complement into a bounded and an unbounded connected component. We call the closure of the bounded component the *cage* of the triple. Of course, the cage has to be filled out by copies of the same tile. We define the *cage number* of a cage as the number of tiles inside the cage, and the cage number of a tiling as the maximal cage number of all cages in the tiling. The three counterexamples correspond to tilings with cage number 1, 2, and 3. We show that no convex tiling with cage number 4 or higher exists.

The situation changes significantly for non-convex monohedral tilings. In that case, non-flag tilings exist for polygons with an arbitrary number of vertices and the cage number can go well beyond 3. As a further contribution, we present a general construction that, for an arbitrary fixed integer c, generates a tiling with cage number c.

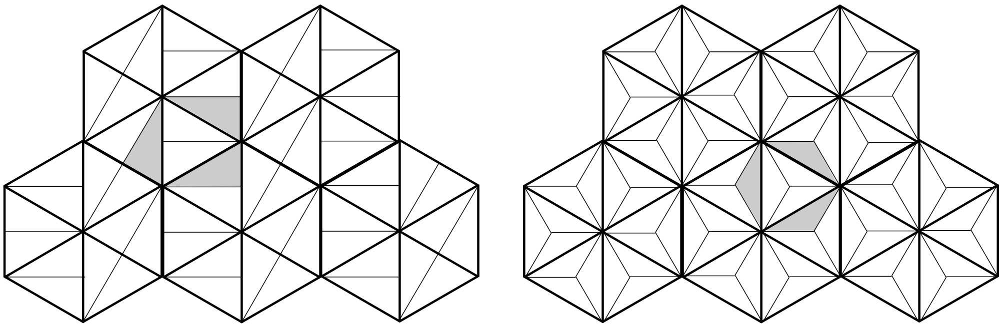

Figure 2 Non-flag Monohedral tilings with cage number 2 (left) and 3 (right). These tilings are obtained from the equilateral tiling from Figure 1 (right) by splitting each triangle in two congruent copies using an altitude, or by splitting each triangle in three congruent copies using the barycenter, respectively. An obstructing triple with the maximal cage number is shaded.

Motivation. The term "flag" originates from the following concepts: A simplicial complex C is called a *flag complex* (also *clique complex*) if it has the following property: if for vertices $\{v_0, \ldots, v_k\}$, all edges (v_i, v_j) are in C, then the k-simplex spanned by $\{v_0, \ldots, v_k\}$ is also in C. Equivalently, C is a flag complex if it is the inclusion-maximal simplicial complex that can be constructed out of the edges of C.

In our setup, a tiling gives rise to a dual simplicial complex, called the *nerve* of the tiling, obtained by defining one vertex per tile, and adding a k-simplex if the corresponding k+1 tiles have a non-empty common intersection. Note that this complex might be high-dimensional – for instance, the nerve of the triangular tiling in Figure 1 contains 5-simplices. The tiling being flag is a necessary condition for the nerve of the tiling being a flag complex. Indeed, if a triple of tiles violates the flag property, the dual complex consists of three edges forming the boundary of a 2-simplex, but the 2-simplex is missing as the three tiles do not commonly intersect. For convex tilings, the tiling is flag if and only if its nerve is a flag complex, which is a simple consequence of Helly's Theorem.

Our question is motivated from an application in computational topology. In [2], the d-dimensional Euclidean space is tiled with permutahedra, and the nerve of a subset of them is the major object of study. In that paper, it is proven (Lemma 10 of [2]) that this nerve is a flag complex (for all d), which simplifies the computation of the complex. The first part of the proof is to show that the tiling has the flag property; for that, two disjoint facets of a permutahedron are considered and it is proven that the neighboring permutahedra along these two facets do not intersect, which implies the flag property. This proof makes use of the special structure of permutahedra and explicitly defines a separating hyperplane for the two neighboring permutahedra, involving lengthy calculations. This note is a first step towards generalizing this useful property of permutahedra to a larger class of tilings, starting with a complete analysis of the planar case.

2 Convex non-flag tilings

We fix a convex monohedral non-flag tiling with an obstructing triple (T_1, T_2, T_3) throughout. Clearly, T_1 (and so, T_2 and T_3) must be a polygon, since any convex non-linear boundary component would require a neighboring tile with a concave boundary component. Since the triple (T_1, T_2, T_3) intersects pairwise, but not commonly, the union $T_1 \cup T_2 \cup T_3$ is a connected set with a hole. While this can also be shown with elementary geometric considerations, a short proof uses the Nerve theorem [1] [4, Ch 4.G], stating that the union of convex shapes is homotopically equivalent (see e.g. [4] for a definition) to their nerve, which in our case is a cycle with three edges. Hence, the union of the three tiles is homotopically equivalent to S^1, a circle.

We call the closure of the (unique) bounded connected component of the complement the *cage* X of the triple. We start with studying the structure of X, relating it with a structure from computational geometry: a *(polygonal) pseudotriangle* is a simple polygon in the plane that is bounded by three concave chains [5]. The degenerate case in which one or several concave chains are just line segments is allowed; hence triangles are a special case of pseudotriangles.

▶ **Lemma 2.1.** *The cage X is a pseudotriangle.*

Proof. The boundary of X consists of boundary curves of the three convex polygons T_1, T_2, and T_3. By convexity, these curves are convex with respect to T_i, and hence concave with respect to the complement. ◀

A pseudotriangle has three *corners* where two concave chains meet. In our case, these corners correspond to intersections of two tiles among $\{T_1, T_2, T_3\}$. The *diameter* of a compact point set is the maximal distance between any pair of points in the set. Two points realizing this distance are called a *diametral pair*. For pseudotriangles, it is easy to see that only corners can form diametral pairs.

▶ **Lemma 2.2.** *Let X be a cage, and let T_X be a tile in the cage. Then, T_X contains two corners of X that form a diametral pair. Moreover, the corresponding concave arc connecting these corners along the boundary of X is a line segment.*

Proof. We define the *latitude* of a compact set S in the plane as the length of the longest line segment that is contained in S. Clearly, congruent sets have the same latitude, and $S' \subseteq S$ implies that the latitude of S' is at most the latitude of S. Let $\ell = \ell(T_1)$ be the latitude of T_1. Then, X must have latitude at least ℓ because it contains at least one congruent copy of T_1.

On the other hand, the latitude of a set is upper bounded by the diameter and for convex sets, both values coincide. Note that for any pair of corners of X, the line segment connecting them is completely contained in some T_i, because the corners are intersection points of tiles. Because all T_i are congruent, the diameter of T_1 is at least the distance of any pair of corners. It follows that the diameter of T_1 is at least the diameter of X. Putting all together, we have

$$\operatorname{diam}(X) \geq \ell(X) \geq \ell(T_1) = \operatorname{diam}(T_1) \geq \operatorname{diam}(X)$$

which implies that all quantities coincide. Since T_X has the same latitude as T_1, it must contain a diametral pair of X, which consists of two corners. Moreover, since T_X is convex, it contains also the line segment between these two corners, implying that X is bounded by this line segment. ◀

Since each tile in a cage has to cover a line segment between two corners, it follows that:

▶ **Corollary 2.3.** *A cage contains at most* 3 *tiles.*

Finally, we can analyze the three possible numbers of tiles inside a cage to show that all of them can only appear for triangular tiles.

▶ **Theorem 2.4.** *If a convex monohedral tiling is not flag, then the tiles are triangles.*

Proof. Assume that tiles (T_1, T_2, T_3) exist that form a cage X. Let c be the number of tiles inside the cage. We know that $c \in \{1, 2, 3\}$ from Corollary 2.3.

If $c = 1$, then X is a tile itself, and hence convex. Because the cage is a pseudotriangle, it is convex if and only if it is a triangle.

If $c = 2$, Lemma 2.2 implies that X has two line segments as sides, and a third concave arc which might be a line segment or a polyline with two segments; a polyline with more vertices is impossible because X is the union of two convex sets. Let v be the corner of X opposite to that third concave arc. Since the two tiles inside the cage intersect in a line segment from v to a point on the opposite arc, the only possibility is that the tiles are triangles.

If $c = 3$, the three tiles inside the cage have to intersect in a common point x as otherwise, they would form a cage again, and X would contain at least 4 tiles. Moreover, by Lemma 2.2, X is a triangle, and each corner is an intersection point of two tiles inside the cage. It follows that the three line segments joining x with the corners of X are the boundaries of the three tiles. However, these line segments split X into three triangles. ◀

We remark that the converse of Theorem 2.4 is not true: there are triangular tilings which are flag (an example can be obtained from the square tiling in Figure 1 (left) by subdividing each square into two triangles arbitrarily). However, the converse becomes true with a further restriction: we call a tiling *face-to-face* if the intersection of two tiles is a facet of both tiles (that is, the tiling carries the structure of a cell complex). For a face-to-face tiling with triangles, it is easy to see that for any triangle T, the three neighboring tiles sharing an edge with T form a cage that contains exactly T. Hence, a planar monohedral face-to-face tiling is flag if and only if the tiles are not triangles.

3 Non-convex tilings

Non-convex monohedral tilings have a long history of research. A remarkable case of instances are *spiral tilings*, for instance the Voderberg tiling[1] or the spiral version of the "Bent Wedge

[1] See `https://en.wikipedia.org/wiki/Voderberg_tiling`

tiling"[2]. By inspecting these tilings, it is not difficult to detect obstructing triples, refuting the possibility that Theorem 2.4 remains true without the convexity assumption.

For an arbitrary integer $n \geq 3$, we describe a construction of a non-convex monohedral tiling with tiles having $2n+1$ vertices such that an obstructing triple with cage number $n-1$ exists. This shows that also Corollary 2.3 is a property that crucially relies on the convexity of the tiles. Our construction is a variant of so-called *radial tilings*[3]. Consider the regular $6n$-gon P inscribed in the unit circle and fix an arbitrary vertex B on that polygon (Figure 3 (left)). Let D be a point on the unit circle such that the triangle OBD is equilateral. In fact, D is a vertex of P. Let c be the circular arc between O and B of the (unit) circle centered at D. Divide c in n sub-arcs of identical length, using $n-1$ additional subdivision points. Let p_1 denote the polyline from O to B defined by these subdivision points.

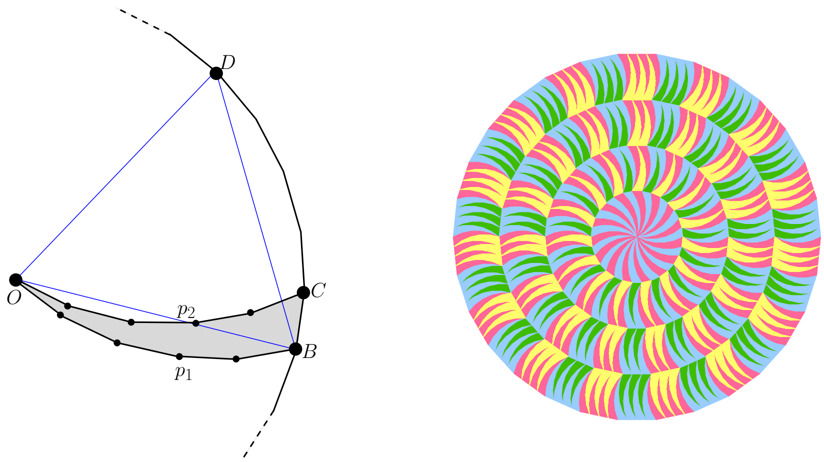

Figure 3 Left: Illustration of the construction of T for $n = 5$. Right: Radial tiling using T.

Next, apply a rotation around the origin (in either direction) by $\frac{2\pi}{6n}$, so that B is mapped to a neighboring vertex C of P. This rotation maps p_1 into a polyline p_2 from O to C. The polygon T bounded by p_1, p_2, and the line segment BC is a polygon with $2n+1$ vertices.

We argue that T indeed admits a monohedral tiling. First of all, by rotating T around the origin by multiples of $\frac{2\pi}{6n}$, $6n$ copies of T cover P. To cover the polygonal annulus between P and $2P$, we observe that the $6n$ reflections of the inner tiles can be completed with $12n$ congruent tiles to fill out the annulus. Extending this idea for the annulus between iP and $(i+1)P$, we can cover the entire plane with copies of T (see Figure 3 (right)).

Finally, to construct a large cage, we modify the tiling inside P: we split the $6n$ tiles into 6 pairwise disjoint groups, each consisting of n consecutive copies of T. Consider such a group G and denote with B and D its two extreme vertices on P. Note that the triangle OBD is equilateral and that the boundary of G consists of three identical polygonal chains (two of them convex and one reflex). It is therefore possible to reflect the whole group G, such that it again covers the same space, and that all tiles in the group intersect at D instead of O. We reflect 3 of the 6 groups inside P, alternating between reflected and unreflected groups. The tiles outside of P are left unchanged. See Figure 4 for two examples. We observe that the cage number of these tilings is $n-1$.

[2] See Steve Dutch's webpage `https://www.uwgb.edu/dutchs/symmetry/radspir1.htm`
[3] See also `https://www.uwgb.edu/dutchs/symmetry/rad-spir.htm`

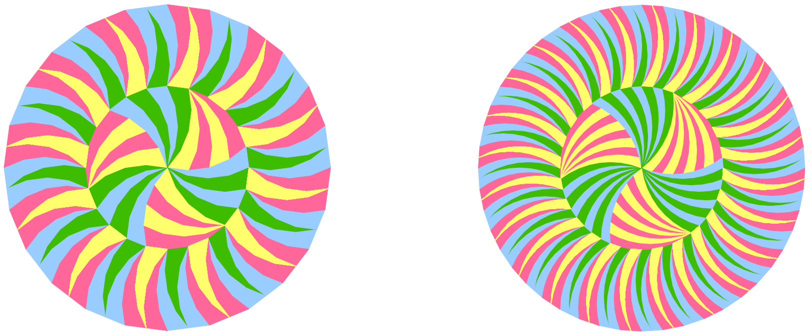

Figure 4 The final outcome of our construction after rearranging the innermost tiles for $n = 4$ (left) and $n = 8$ (right). In both cases, there are 6 groups of tiles around the origin, and three of them are rotated. The tile of a rotated group at the boundary of the $6n$-gon together with the extremal tiles of the neighboring (unrotated) groups form an obstructing triple with cage number 3 on the left, and 7 on the right.

4 Conclusion

Various questions remain open for the non-convex case. For instance: is there a *monohedral* tiling that is flag such that its nerve is not a flag complex? While it is rather simple to give an example of four non-convex shapes whose nerve is the boundary of a tetrahedron, it is not so simple to provide such an example with congruent shapes, and even less so to construct such a scenario in a monohedral tiling. Another question is what would be the maximal cage number possible for a monohedral tiling with a k-vertex polygon. Our paper establishes the lower bound of $\frac{k-3}{2}$. We are currently not able to provide any upper bound.

More in line with our original motivation, we plan to investigate convex monohedral tilings in higher dimension next. In detail, we want to characterize large classes of such tilings for which the nerve is a flag complex. Already in three dimensions, the natural generalization of Theorem 2.4 that all non-tetrahedral tilings have this property fails because we can simply extend Figure 1 (right) to the third dimension using triangular prisms. A statement in reach seems to be the following: restricting to face-to-face tilings, we call a tiling in $\mathbb{R}^d$ *generic* if at most $d + 1$ tiles meet in a common point. We claim that the nerve of a generic tiling is a flag complex. This would include the permutahedral scenario considered in [2].

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