Stabbing Pairwise Intersecting Disks by Five Points*

Sariel Har-Peled[1], Haim Kaplan[2], Wolfgang Mulzer[3], Liam Roditty[4], Paul Seiferth[3], Micha Sharir[2], and Max Willert[3]

1 **Department of Computer Science, University of Illinois, Urbana, USA**
`sariel@illinois.edu`

2 **School of Computer Science, Tel Aviv University, Israel**
`{haimk,michas}@post.tau.ac.il`

3 **Department of Computer Science, Freie Universität Berlin, Germany**
`{mulzer,pseiferth,willerma}@inf.fu-berlin.de`

4 **Department of Computer Science, Bar Ilan University, Israel**
`liamr@macs.biu.ac.il`

Abstract

We present a deterministic linear time algorithm to find a set of five points that stab a set of n pairwise intersecting disks in the plane. We also give a simple construction with 13 pairwise intersecting disks that cannot be stabbed by three points.

1 Introduction

Let $\mathcal{D}$ be a set of n pairwise intersecting disks in the plane. If every three disks in $\mathcal{D}$ have a nonempty intersection, then, by Helly's theorem, the whole intersection $\cap\mathcal{D}$ is nonempty [6–8]. Thus, $\mathcal{D}$ can be *stabbed* by one point. More generally, when there are three disks with empty intersection, $\mathcal{D}$ can still always be stabbed by four points. In July 1956, Danzer presented a proof at Oberwolfach (see [3]). Since Danzer was not satisfied with his proof, he never published it, but he gave a new proof in 1986 [3]. Previously, in 1981, Stachó published a proof for the existence of four stabbing points [11], using similar arguments as in his previous construction of five stabbing points [10]. Hadwiger and Debrunner showed that three points suffice for unit disks [5]. Danzer's upper bound proof is fairly involved, and there seems to be no obvious way to turn it into an efficient algorithm. The two constructions of Stachó are simpler, but not enough for an easy subquadratic algorithm. We present a new argument that yields five stabbing points. Our proof is constructive and allows us to find the stabbing points deterministically in linear time.

As for lower bounds, Grünbaum gave an example of 21 pairwise intersecting disks that cannot be stabbed by three points [4]. Later, Danzer reduced the number of disks to ten [3]. This example is close to optimal, because every set of eight disks can be stabbed by three points [10]. It is hard to verify the lower bound by Danzer for ten disks—even with dynamic geometry software. We present a simple construction that uses 13 disks.

2 The geometry of pairwise intersecting disks

Let $\mathcal{D}$ be a set of n pairwise intersecting disks. A disk $D_i \in \mathcal{D}$ is given by its center c_i and its radius r_i. We assume without loss of generality that no disk is contained in another. The *lens* of two disks $D_i, D_j \in \mathcal{D}$ is the set $L_{i,j} = D_i \cap D_j$. Let u be any of the two intersection points of ∂D_i

* Supported in part by grant 1367/2016 from the German-Israeli Science Foundation (GIF). W.M. supported in part by ERC StG 757609. P.S. and W.M. supported in part by DFG grant MU-3501/1.

34th European Workshop on Computational Geometry, Berlin, Germany, March 21–23, 2018.

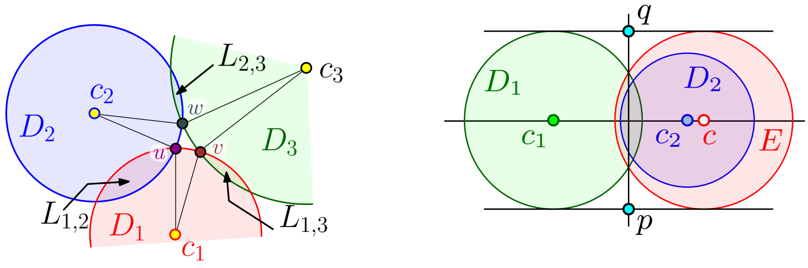

Figure 1 Left: At least one lens angle is large. **Right:** D_1 and E have the same radii and lens angle $2\pi/3$. By Lemma 2.2, D_2 is a subset of E. $\{c_1, c, p, q\}$ is the set P from Lemma 2.4.

and ∂D_j. The angle $\angle c_i u c_j$ is called the *lens angle* of D_i and D_j. It is at most π. Three disks D_i, D_j, and D_k are *non-Helly* if $D_i \cap D_j \cap D_k = \emptyset$. We present some useful geometric lemmas.

▶ **Lemma 2.1.** *Among any three non-Helly pairwise intersecting disks D_1, D_2, and D_3, there are two disks with lens angle larger than $2\pi/3$.*

Proof. By assumption, the lenses $L_{1,2}$, $L_{1,3}$ and $L_{2,3}$ are pairwise disjoint. Let u be the vertex of $L_{1,2}$ nearer to D_3, and let v, w be the analogous vertices of $L_{1,3}$ and $L_{2,3}$ (see Figure 1, Left). Consider the simple hexagon $c_1 u c_2 w c_3 v$, and write $\angle u$, $\angle v$, and $\angle w$ for the interior angles at u, v, and w. The sum of all interior angles is 4π. Thus, $\angle u + \angle v + \angle w < 4\pi$, so at least one angle is less than $4\pi/3$. It follows that the exterior angle at u, v, or w must be larger than $2\pi/3$. ◀

▶ **Lemma 2.2.** *Let D_1 and D_2 be two intersecting disks with radii $r_1 \geq r_2$ and lens angle $\alpha \geq 2\pi/3$. Let E be the unique disk with radius r_1 and center c, such that (i) the centers c_1, c_2, and c are collinear and c lies on the same side of c_1 as c_2; and (ii) the lens angle of D_1 and E is exactly $2\pi/3$ (see Figure 1, Right). Then, if c_2 lies between c_1 and c, we have $D_2 \subseteq E$.*

Proof. Let $x \in D_2$. Then, since c_2 lies between c_1 and c, the triangle inequality gives

$$|xc| \leq |xc_2| + |c_2 c| = |xc_2| + |c_1 c| - |c_1 c_2|. \tag{1}$$

Since $x \in D_2$, we get $|xc_2| \leq r_2$. Also, since D_1 and E each have radius r_1 and form the lens angle $2\pi/3$, it follows that $|c_1 c| = \sqrt{3} r_1$. Finally, by the law of cosines, $|c_1 c_2| = \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 \cos\alpha}$. As $\alpha \geq 2\pi/3$ and $r_1 \geq r_2$, we get $\cos\alpha \leq -0.5 \leq (\sqrt{3} - 1.5)\frac{r_1}{r_2} - \sqrt{3} + 1$, so

$$|c_1 c_2|^2 = r_1^2 + r_2^2 - 2 r_1 r_2 \cos\alpha \geq r_1^2 + r_2^2 - 2 r_1 r_2 \left(\left(\sqrt{3} - 1.5\right) \frac{r_1}{r_2} - \sqrt{3} + 1 \right) = \left(r_1 \left(\sqrt{3} - 1\right) + r_2 \right)^2.$$

Plugging this into Eq. (1), we obtain $|xc| \leq r_2 + \sqrt{3} r_1 - \left(r_1 \left(\sqrt{3} - 1\right) + r_2 \right) = r_1$, i.e., $x \in E$. ◀

▶ **Lemma 2.3.** *Let D_1 and D_2 be two intersecting disks of equal radius r with lens angle $2\pi/3$. There is a set P of four points so that any disk F of radius at least r that intersects both D_1 and D_2 contains a point of P.*

Proof. Consider the two tangent lines of D_1 and D_2, and let p and q be the midpoints on these lines between the respective two tangency points. We set $P = \{c_1, c_2, p, q\}$ (see Figure 2, Left).

Given F, we decrease its radius, keeping its center fixed, until either the radius becomes r or until F is tangent to D_1 or D_2. Suppose the latter case holds and F is tangent to D_1. We move the center of F continuously along the line spanned by the center of F and c_1 towards c_1, decreasing the radius of F to maintain the tangency. We stop when either the radius of F reaches

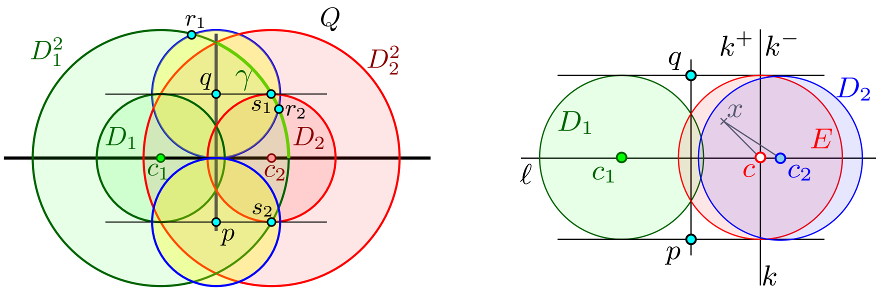

Figure 2 Left: $P = \{c_1, c_2, p, q\}$ is the stabbing set. The green arc $\gamma = \partial(D_1^2 \cap D_2^2) \cap Q$ is covered by $D_1^2 \cap D_q$. **Right:** Situation (ii) in the proof of Lemma 2.4: $D_2 \not\subseteq E$. x is an arbitrary point in $D_2 \cap F \cap k^+$. The angle at c in the triangle Δxcc_2 is $\geq \pi/2$.

r or F becomes tangent to D_2. We obtain a disk $G \subseteq F$ with center $c = (c_x, c_y)$ so that either: (i) radius$(G) = r$ and G intersects both D_1 and D_2, or (ii) radius$(G) \geq r$ and G is tangent to both D_1 and D_2. Since $G \subseteq F$, it suffices to show that $G \cap P \neq \emptyset$. We introduce a coordinate system, setting the origin o midway between c_1 and c_2, so that the y-axis passes through p and q. Then, in the manner depicted in Figure 2 (left), we have $c_1 = (-\sqrt{3}r/2, 0)$, $c_2 = (\sqrt{3}r/2, 0)$, $q = (0, r)$, and $p = (0, -r)$.

For case (i), let D_1^2 be the disk of radius $2r$ centered at c_1, and D_2^2 the disk of radius $2r$ centered at c_2. Since G has radius r and intersects both D_1 and D_2, its center c has distance at most $2r$ from both c_1 and c_2, i.e., $c \in D_1^2 \cap D_2^2$. Let D_p and D_q be the two disks of radius r centered at p and q. We will show that $D_1^2 \cap D_2^2 \subseteq D_1 \cup D_2 \cup D_p \cup D_q$. Then it is immediate that $G \cap P \neq \emptyset$. By symmetry, it is enough to focus on the upper-right quadrant $Q = \{(x, y) \mid x \geq 0, y \geq 0\}$. We show that all points in $D_1^2 \cap Q$ are covered by $D_2 \cup D_q$. Without loss of generality, we assume that $r = 1$. Then, the two intersection points of D_1^2 and D_q are $r_1 = (\frac{5\sqrt{3}-2\sqrt{87}}{28}, \frac{38+3\sqrt{29}}{28}) \approx (-0.36, 1.93)$ and $r_2 = (\frac{5\sqrt{3}+2\sqrt{87}}{28}, \frac{38-3\sqrt{29}}{28}) \approx (0.98, 0.78)$, and the two intersection points of D_1^2 and D_2 are $s_1 = (\frac{\sqrt{3}}{2}, 1) \approx (0.87, 1)$ and $s_2 = (\frac{\sqrt{3}}{2}, -1) \approx (0.87, -1)$. Let γ be the boundary curve of D_1^2 in Q. Since $r_1, s_2 \notin Q$ and since $r_2 \in D_2$ and $s_1 \in D_q$, it follows that γ does not intersect the boundary of $D_2 \cup D_q$ and hence $\gamma \subset D_2 \cup D_q$. Furthermore, the subsegment of the y-axis from o to the startpoint of γ is contained in D_q, and the subsegment of the x-axis from o to the endpoint of γ is contained in D_2. Hence, the boundary of $D_1^2 \cap Q$ lies completely in $D_2 \cup D_q$, and since $D_2 \cup D_q$ is simply connected, it follows that $D_1^2 \cap Q \subseteq D_2 \cup D_q$, as desired.

For case (ii), since G is tangent to D_1 and D_2, the center c of G is on the perpendicular bisector of c_1 and c_2, so the points p, o, q and c are collinear. Suppose without loss of generality that $c_y \geq 0$. Then, it is easily checked that c lies above q, and radius$(G) + r = |c_1c| \geq |oc| = r + |qc|$, so $q \in G$. ◀

▶ **Lemma 2.4.** *Consider two intersecting disks D_1 and D_2 with radii $r_1 \geq r_2$, having lens angle at least $2\pi/3$. Then, there is a set P of four points such that any disk F of radius at least r_1 that intersects both D_1 and D_2 contains a point of P.*

Proof. Let ℓ be the line through c_1 and c_2. Let E be the disk of radius r_1 and center $c \in \ell$ that satisfies the conditions (i) and (ii) of Lemma 2.2. Let P be the point set $\{c_1, c, p, q\}$ specified in the proof of Lemma 2.3, with respect to D_1 and E (see Figure 1, Right). We claim that

$$D_1 \cap F \neq \emptyset \;\wedge\; D_2 \cap F \neq \emptyset \;\Rightarrow\; E \cap F \neq \emptyset. \quad (*)$$

Once (*) is established, we are done by Lemma 2.3. If $D_2 \subseteq E$, then (*) is immediate, so assume that $D_2 \not\subseteq E$. By Lemma 2.2, c lies between c_1 and c_2. Let k be the line through c perpendicular to ℓ, and let k^+ be the open halfplane bounded by k with $c_1 \in k^+$ and k^- the open halfplane bounded by k with $c_1 \notin k^-$. Since $|c_1 c| = \sqrt{3} r_1 > r_1$, we have $D_1 \subset k^+$ (see Figure 2, Right). Recall that F has radius at least r_1 and intersects D_1 and D_2. We distinguish two cases: (i) there is no intersection of F and D_2 in k^+, and (ii) there is an intersection of F and D_2 in k^+.

For case (i), let x be any point in $D_1 \cap F$. Since we know that $D_1 \subset k^+$, we have $x \in k^+$. Moreover, let y be any point in $D_2 \cap F$. By assumption (i), y is not in k^+, but it must be in the infinite strip defined by the two tangents of D_1 and E. Thus, the line segment $\overline{xy}$ intersects the diameter segment $k \cap E$. Since F is convex, the intersection of $\overline{xy}$ and $k \cap E$ is in F, so $E \cap F \neq \emptyset$.

For case (ii), let x be any point in $D_2 \cap F \cap k^+$. Consider the triangle $\Delta x c c_2$. Since $x \in k^+$, the angle at c is at least $\pi/2$ (Figure 2, Right). Thus, $|xc| \leq |xc_2|$. Moreover, since $x \in D_2$, we know that $|xc_2| \leq r_2 \leq r_1$. Hence, we have $|xc| \leq r_1$ so $x \in E$ and (*) follows, as $x \in E \cap F$. ◀

3 Stabbing disks in linear time

Let $\mathcal{D}$ be a set of n pairwise intersecting disks. For $r > 0$, we define $\bigcap_{\leq r} \mathcal{D}$ to be the intersection of all disks in $\mathcal{D}$ with radius at most r. The set $\bigcap_{<r} \mathcal{D}$ is defined analogously. Moreover, let X be a non-empty intersection of finitely many disks. Then, $\mathcal{V}(X)$ is the set of vertices on the boundary of X.

▶ **Lemma 3.1.** *For a set $\mathcal{D}$ of n pairwise intersecting disks, we can decide in linear time if the intersection $\bigcap \mathcal{D}$ is empty. In the same time, we can compute a point in $\bigcap \mathcal{D}$, if it exists, or a non-Helly triple D_i, D_j, D_k with $r_i, r_j \leq r_k$, such that $\bigcap_{<r_k} \mathcal{D} \neq \emptyset$, otherwise.*

Proof. Consider a subset $\mathcal{D}'$ of $\mathcal{D}$ and assume first that $\bigcap \mathcal{D}' = \emptyset$. In this case, there exists a disk $D_k \in \mathcal{D}'$ with radius r_k such that $\bigcap_{<r_k} \mathcal{D}' \neq \emptyset$ and $\bigcap_{\leq r_k} \mathcal{D}' = \emptyset$. We set $\mathrm{ind}(\mathcal{D}') = k$ and $\mathrm{rad}(\mathcal{D}') = r_k$. Next, assume that $\bigcap \mathcal{D}' \neq \emptyset$. In this case, we set $\mathrm{ind}(\mathcal{D}') = \infty$ and $\mathrm{rad}(\mathcal{D}') = \infty$. Now, for $\mathcal{D}' \subseteq \mathcal{D}$, we define $w(\mathcal{D}') = \Big(\mathrm{rad}(\mathcal{D}'), -\min\big\{d(v, D_{\mathrm{ind}(\mathcal{D}')}) \mid v \in \mathcal{V}(\bigcap_{<\mathrm{rad}(\mathcal{D}')} \mathcal{D}')\big\}\Big)$. If $\mathrm{ind}(\mathcal{D}') = \infty$ we set $d(v, D_\infty) = v_y$, the y-coordinate of v. Chan has observed that the problem $(\mathcal{D}, w)$ is *LP-type* [1,9]. The combinatorial dimension of $(\mathcal{D}, w)$ is 3, and therefore, the violation test can be done in constant time. Furthermore, for a basis $\mathcal{B}$ of $(\mathcal{D}, w)$, let $\mathrm{vio}(\mathcal{B})$ be the set of disks in $\mathcal{D}$ that violate $\mathcal{B}$, i.e., for all $D \in \mathrm{vio}(\mathcal{B})$, we have $w(\mathcal{B} \cup \{D\}) < w(\mathcal{B})$. Then, $(\mathcal{D}, \mathcal{R} = \{\mathrm{vio}(\mathcal{B}) \mid \mathcal{B} \text{ is a basis in } \mathcal{D}\})$ is the underlying range space for the LP-type problem, and one can check that it has constant VC-dimension. Thus, we can use the deterministic algorithm by Chazelle and Matoušek [2] to compute $w(\mathcal{D})$ and a corresponding basis $\mathcal{B}$ in $O(n)$ time. One can show that $\mathcal{B}$ is either a non-Helly triple for $\mathcal{D}$ with the desired properties, or that $\mathcal{B}$ yields a stabbing point for $\mathcal{D}$. ◀

▶ **Theorem 3.2.** *Given a set $\mathcal{D}$ of n pairwise intersecting disks in the plane, we can find in linear time a set S of five points such that every disk of $\mathcal{D}$ contains at least one point of S.*

Proof. Using Lemma 3.1, we decide if $\cap \mathcal{D}$ is empty. If not, we return a point in the common intersection. Otherwise, the lemma gives us a non-Helly tripe with smallest maximum radius r_k. For the disks $D_\ell \in \mathcal{D}$ with $r_\ell < r_k$, we can obtain in linear time a stabbing point s by using Helly's theorem and Lemma 3.1. Next, by Lemma 2.1, there are two disks D' and D'' among D_i, D_j and D_k whose lens angle is at least $2\pi/3$. Let P be the set of four points, as described in the proof of Lemma 2.4, that stabs any disk of radius at least r_k that intersects both D' and D''. Then $S = \{s\} \cup P$ is a set of five points that stabs all disks of $\mathcal{D}$. ◀

4 13 pairwise intersecting disks that cannot be stabbed by 3 points

The construction begins with an inner disk A, say of radius 1, and three larger disks D_1, D_2, D_3 of equal size, so that A is tangent to all three disks, and each pair of the disks are tangent to each other. Denote the contact point of A and D_i by ξ_i, for $i = 1, 2, 3$.

We add six very large disks as follows. For $i = 1, 2, 3$, we draw the two common outer tangents to A and D_i, and denote by T_i^- and T_i^+ the halfplanes that are bounded by these tangents and are openly disjoint from A. For concreteness, the labels T_i^- and T_i^+ are such that the points of tangency between A and T_i^-, D_i, and T_i^+, appear along ∂A in this counterclockwise order. One can show that the nine points of tangency between A and the other disks and halfplanes are all distinct (see Figure 3). We regard the six halfplanes T_i^-, T_i^+, for $i = 1, 2, 3$, as disks; in the end, we can apply a suitable inversion to turn the disks and halfplanes into actual disks, if so desired.

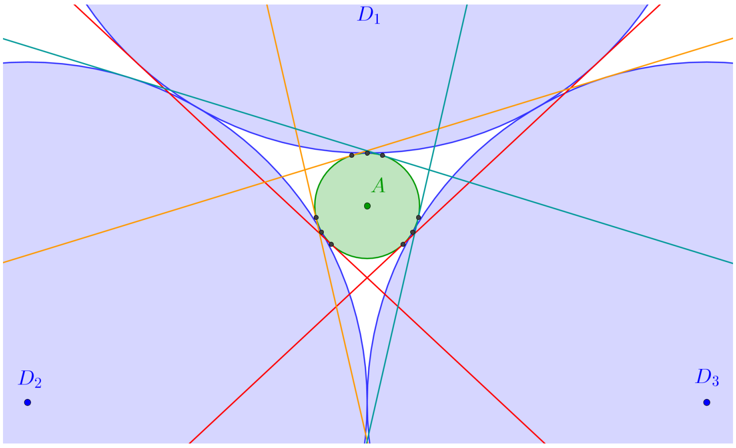

Figure 3 Each common tangent ℓ represents a very large disk tangent to the disks to which ℓ is tangent. The nine points of tangency are all distinct.

Finally, we construct three additional disks A_1, A_2, A_3. To construct A_i, we slightly expand A into a disk A'_i of radius $1 + \varepsilon_1$, while keeping it touching D_i at ξ_i. We then roll A'_i clockwise along D_i, by a tiny angle $\varepsilon_2 \ll \varepsilon_1$ to obtain A_i.

This completes the construction, giving 13 disks. For sufficiently small ε_1 and ε_2, we can ensure the following properties for each A_i: (i) A_i intersects all other 12 disks, (ii) the nine intersection regions $A_i \cap D_j$, $A_i \cap T_j^-$, $A_i \cap T_j^+$, for $j = 1, 2, 3$, are pairwise disjoint. (iii) $\xi_i \notin A_i$.

▶ **Lemma 4.1.** *The* 13 *disks in the construction cannot be stabbed by three points.*

Proof. Consider any set of three points and suppose they form a stabbing set. Let A^* be the union $A \cup A_1 \cup A_2 \cup A_3$. If p is a stabbing point in A^*, then typically p will stab all these four disks (unless p lies at certain peculiar locations), but, by construction, it stabs at most one of the nine remaining disks. It is thus impossible for all three stabbing points to lie in A^*, but at least one of them must lie there.

Assume first that A^* contains two stabbing points. As just argued, there are at most two of the remaining disks that are stabbed by these points. The following cases can then arise.

(a) The stabbed disks are both halfplanes. Then D_1, D_2, D_3 form a non-Helly triple, i.e. they do not have a common intersection, and none of them is stabbed. Since a non-Helly triple must be stabbed by at least two points, an unstabbed disk remains.

(b) The stabbed disks are both among D_1, D_2, D_3. Then the six unstabbed halfplanes form many non-Helly triples [1], e.g., T_1^-, T_2^-, and T_3^-, and again a disk remains unstabbed.

(c) One stabbed disk is D_1, D_2, or D_3, and the other is a halfplane. Then, there is (at least) one disk D_i such that it, and the two associated halfplanes T_i^-, T_i^+ are all unstabbed. (D_i is

[1] Note that it is easy to extend the definition of non-Helly triples to halfplanes.

a disk that is not stabbed by either of the two initial points, and neither of its two tangent halfplanes is stabbed.) Then D_i, T_i^-, and T_i^+ form an unstabbed non-Helly triple.

Assume then that A^* contains only one stabbing point p, so at most one of the nine remaining disks is stabbed by p. Since p is the only point that stabs all three disks A_1, A_2, A_3, it cannot be any of ξ_1, ξ_2, ξ_3, so the other disk that it stabs (if there is such a disk) must be a halfplane. That is, p does not stab any of D_1, D_2, D_3. Since D_1, D_2, D_3 form a non-Helly triple, they require two points to stab them all. Moreover, since we only have two points at our disposal, one of them must be the point of tangency of two of these disks, say of D_2 and D_3. This point stabs only two of the six halfplanes (concretely, they are T_1^- and T_1^+). But then D_1, T_2^+, and T_3^- form an unstabbed non-Helly triple. ◀

5 Conclusion

We presented a simple algorithm for the computation of five stabbing points for a set of pairwise intersecting disks by solving a corresponding LP-type problem. Nevertheless, the question remains open how to use the proofs of Danzer or Stachó (or any other technique) for an efficient construction of four stabbing points. Since eight disks can always be stabbed by three points [10], it remains open whether nine disks can be stabbed by three points or not. Furthermore, it would be interesting to find a simpler construction of ten pairwise intersecting disks that cannot be stabbed by three points.

Acknowledgments. We would like to thank Timothy Chan for pointing us to the direct LP-type algorithm described in Lemma 3.1.

References

1. T. M. Chan. An optimal randomized algorithm for maximum Tukey depth. In *Proceedings of the fifteenth annual ACM-SIAM Symposium on Discrete Algorithms*, pages 430–436, 2004.
2. B. Chazelle and J. Matoušek. On linear-time deterministic algorithms for optimization problems in fixed dimension. *Journal of Algorithms*, 21(3):579–597, 1996.
3. L. Danzer. Zur Lösung des Gallaischen Problems über Kreisscheiben in der Euklidischen Ebene. *Studia Sci. Math. Hungar.*, 21(1-2):111–134, 1986.
4. B. Grünbaum. On intersections of similar sets. *Portugal. Math.*, 18:155–164, 1959.
5. H. Hadwiger and H. Debrunner. Ausgewählte Einzelprobleme der kombinatorischen Geometrie in der Ebene. *Enseignement Math. (2)*, 1:56–89, 1955.
6. E. Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 32:175–176, 1923.
7. E. Helly. Über Systeme von abgeschlossenen Mengen mit gemeinschaftlichen Punkten. *Monatshefte für Mathematik*, 37(1):281–302, 1930.
8. J. Radon. Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten. *Mathematische Annalen*, 83(1):113–115, 1921.
9. M. Sharir and E. Welzl. A combinatorial bound for linear programming and related problems. *STACS 92*, pages 567–579, 1992.
10. L. Stachó. Über ein Problem für Kreisscheibenfamilien. *Acta Sci. Math. (Szeged)*, 26:273–282, 1965.
11. L. Stachó. A solution of Gallai's problem on pinning down circles. *Mat. Lapok*, 32(1-3):19–47, 1981/84.