Augmenting a tree to a k-arbor-connected graph with pagenumber k

Toru Hasunuma[1]

1 **Tokushima University**
hasunuma@tokushima-u.ac.jp

Abstract

A tree is one of the most fundamental structures of graphs and has good properties on layouts, while it is weak from a fault-tolerant point of view. Motivated by these points of view, we consider an augmentation problem for a tree to increase fault-tolerance while preserving its good property on book-embeddings. A k-arbor-connected graph is a graph which has k spanning trees such that for any two vertices, the k paths between them in the k spanning trees are pairwise edge-disjoint and internally vertex-disjoint. We show that any tree with n vertices can be augmented in $O(nk)$ time to a minimum k-arbor-connected graph with pagenumber k for any k at most the radius of the tree. Our result is optimal for both the number of added edges and the number of pages for a book-embedding of a resultant graph. Besides, we extend our augmentation for trees to cacti.

1 Introduction

Throughout the paper, a graph means a simple undirected graph. Let $G = (V, E)$ be a graph. An augmentation problem for a graph is to find a set E' of pairs of non-adjacent vertices in G such that the augmented graph $G' = (V, E \cup E')$ satisfies a given condition.

A *book* is a structure consisting of a line called the *spine* and half planes called *pages* sharing the spine as a common boundary. A *k-page book-embedding* of G is defined by an assignment of the vertices of G to distinct points on the spine, i.e., a vertex-ordering σ of $V(G)$, and an assignment of the edges of G to pages such that no two edges assigned to the same page cross, where two edges uv and xy cross under σ if $\sigma(u) < \sigma(x) < \sigma(v) < \sigma(y)$. The *pagenumber* $\mathrm{pn}(G)$ of G is the minimum number of pages for a book-embedding of G. Book-embeddings have applications to VLSI layouts and there are many results on the subject until now (e.g., see [1, 2, 4]). In particular, one of the most famous results on book-embeddings is that every planar graph can be embedded in 4 pages [15].

Let $T_1, T_2, \ldots, T_k$ be spanning trees in G. If for any two vertices of G, the k paths between them in $T_1, T_2, \ldots, T_k$ are pairwise edge-disjoint and internally vertex disjoint, then $T_1, T_2, \ldots, T_k$ are *completely independent spanning trees* in G. Completely independent spanning trees can be applied to fault-tolerant communication problems, e.g., fault-tolerant broadcasting problems, since by deleting any $k-1$ vertices, at least one of the k completely independent spanning trees keeps its connectedness. We define the *arbor-connectivity* of G as the maximum number $\tau(G)$ of completely independent spanning trees in G, and G is *k-arbor-connected* if $\tau(G) \geq k$. So far, arbor-connectedness of graphs has been studied for graph classes related to interconnection networks (e.g., see [3, 5, 12]). It has also been shown that every maximal 4-connected planar graph is 2-arbor-connected [7], and G is $\lfloor \frac{n}{k} \rfloor$-arbor-connected if the minimum degree of G is at least $n - k$ where $3 \leq k \leq \frac{n}{2}$ [8]. Although any k-arbor-connected graph is k-vertex-connected, it has been proved [13] that there is no direct relationship between the vertex-connectivity and the arbor-connectivity; for any $k \geq 2$, there exists a k-vertex-connected graph G with $\tau(G) = 1$. From an algorithmic point of view, it has been shown that the problem of deciding whether a given graph is 2-arbor-connected is NP-complete [7].

34th European Workshop on Computational Geometry, Berlin, Germany, March 21–23, 2018.

A tree is one of the most fundamental structures of graphs and has good properties on layouts, e.g., the pagenumber of a tree is one. On the other hand, a tree is weak from a fault-tolerant point of view since it can be disconnected by deleting only one vertex. Motivated by these points of view, we consider an augmentation problem for a tree to increase fault-tolerance while preserving its good property on book-embeddings. On connectivity augmentation of graphs with geometric constraints, there are many results until now (see [9]). In particular, Kant and Bodlaender [10] have shown that the planarity-preserving minimum 2-vertex-connectivity augmentation problem is NP-hard, and Rutter and Wolff [14] have proved that the corresponding 2-edge-connectivity version is also NP-hard.

In this paper, we show that any tree T can be augmented in $O(nk)$ time to a minimum k-arbor-connected graph T^* which can be embedded in k pages for any k at most the radius of T. Every graph with n vertices and m edges needs at least $\lceil \frac{m-n}{n-3} \rceil$ pages for its book-embedding, which follows from the fact that a graph with pagenumber one is outerplanar [2]. This means that any k-arbor-connected graph cannot be embedded in $k-1$ pages, i.e., the pagenumber of T^* is determined to be k. Thus, our augmentation result is optimal for both the number of added edges and the number of pages for a book-embedding of a resultant augmented graph. Our augmented graph T^* also has a property that T^* is decomposed into completely independent spanning trees $T_1, T_2, \ldots, T_k$ such that each T_i can be embedded in one page under the same vertex-ordering. Besides, we extend our augmentation for trees to cacti and present an augmentation result for cycles.

2 Preliminaries

Given a set F of edges, the graph induced by F is denoted by $\langle F \rangle$, i.e., $V(\langle F \rangle) = \{u \mid uv \in F\}$ and $E(\langle F \rangle) = F$. The *distance* $d_G(u,v)$ of vertices u and v in a connected graph G is the length of a shortest path between u and v. The *eccentricity* $e_G(w)$ of a vertex w in G is $\max_{v \in V(G)} d_G(w,v)$. The *diameter* $\mathrm{diam}(G)$ of G is $\max_{w \in V(G)} e_G(w)$ and the *radius* $\mathrm{rad}(G)$ of G is $\min_{w \in V(G)} e_G(w)$. A *central vertex* of G is a vertex v with $e_G(v) = \mathrm{rad}(G)$. The *center* of G is the set of central vertices of G. Let T be a tree rooted at a vertex r. The *ℓ-ancestor* $p_\ell(v)$ of a vertex v in T is a vertex w which is on the path from r to v such that $d_T(v,w) = \ell$. If w is the ℓ-ancestor of v, then v is an *ℓ-descendant* of w. The set of ℓ-descendants of w is denoted by $D_\ell(w)$. The *lowest common ancestor* $\mathrm{lca}_T(u,v)$ of u and v in T is a common ancestor w of u and v in T such that there is no descendant of w which is a common ancestor of u and v. The *height* $h(T)$ of T is $\max_{v \in V(T)} d_T(r,v)$. A *leaf* of T is a vertex with degree one, while an *internal vertex* of T is a vertex with degree greater than one. The set of internal vertices in T is denoted by $V_I(T)$. A *star* is a tree in which there exists at most one internal vertex. A *cut-vertex* of G is a vertex v such that the graph obtained from G by deleting v is disconnected. A *block* of G is a maximal subgraph of G without a cut-vertex. A *cactus* is a graph whose every block is either a cycle or the complete graph with two vertices. A *cycle edge* of a cactus is an edge on a cycle. A *unicyclic graph* G is a graph with exactly one cycle and the cycle is denoted by $C(G)$.

Let σ be a vertex-ordering of G, i.e., a bijection from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$. When $\sigma(u) < \sigma(v)$, we simply write $u <_\sigma v$. For $u, v \in S \subseteq V(G)$, if $u <_\sigma v$ such that there is no vertex $w \in S$ with $u <_\sigma w <_\sigma v$, then u and v are *consecutive* in S under σ and we write $u \lessdot_{\sigma,S} v$. When $S = V(G)$, we may write $u \lessdot_\sigma v$. In order to construct completely independent spanning trees, we use a characterization shown in [6]; spanning trees $T_1, T_2, \ldots, T_k$ in G are completely independent if and only if $E(T_i) \cap E(T_j) = \emptyset$ and $V_I(T_i) \cap V_I(T_j) = \emptyset$ for any $1 \le i < j \le k$.

3 Results

▶ **Theorem 3.1.** *Any tree T with n vertices can be augmented to a minimum k-arbor-connected graph with pagenumber k for any $2 \leq k \leq \mathrm{rad}(T)$ in $O(nk)$ time.*

Proof. If T has two central vertices, then let x and y be the central vertices of T. Note that $xy \in E(T)$. Otherwise, let x be the central vertex of T and let y a vertex adjacent to x such that y is on a path between x and a vertex v with $d_T(x,v) = \mathrm{rad}(T)$. Let T^+ be the tree obtained from T by adding a new vertex z, joining it to x and y, and deleting the edge xy. In what follows, ancestors and descendants of a vertex are defined based on T^+ rooted at z unless otherwise stated. For any vertex u in T^+, T_u denotes the subtree rooted at u in T^+. By the definitions of x and y, it holds that $h(T_x) = \mathrm{rad}(T) \geq h(T_y) \geq \mathrm{rad}(T) - 1$.

Regarding the vertex z as the root of T^+, compute a depth-first-search ordering $\sigma^+ : V(T^+) \mapsto \{1, 2, \ldots, n+1\}$, where $\sigma^+(z) = 1$. Then, let $\sigma : V(T) \mapsto \{1, 2, \ldots, n\}$ be the vertex-ordering of T defined to be $\sigma(v) = \sigma^+(v) - 1$. Now let $V_i = D_{i+1}(z)$ for $0 \leq i < \mathrm{rad}(T)$. Also, let $W_\ell = \bigcup_{i \bmod k = \ell} V_i$ for each $0 \leq \ell < k$. We first divide $E(T) - \{xy\}$ into k subsets $E_1, E_2, \ldots, E_k$ defined as follows: for each $1 \leq i \leq k$,

- $E_i = \{vw \mid v \in W_{i-1},\ w \in D_1(v)\}$.

The set of added edges in our augmentation is divided into three types defined as follows: for each $1 \leq i \leq k$,

- $A_i = \{vw \mid v \in W_{i-1},\ w \in D_j(v),\ 2 \leq j \leq k\}$,
- $B_i = \{uw \mid u, v \in V_{i-1},\ u \lessdot_{\sigma, V_{i-1}} v,\ \sigma^{-1}(\max_{u' \in V(T_u)} \sigma(u')) <_\sigma w \leq_\sigma v\}$,
- $B'_i = \{wv \mid v = \sigma^{-1}(\max_{u' \in V_{i-1}} \sigma(u')),$
  $w <_\sigma \sigma^{-1}(\min_{u' \in V_{i-1}} \sigma(u'))$ or $\sigma^{-1}(\max_{u' \in V(T_v)} \sigma(u')) <_\sigma w\}$.

Note that $B_1 = \{xy\}$. Based on these sets, we define $T_1, T_2, \ldots, T_k$ as $T_i = \langle E_i \cup A_i \cup B_i \cup B'_i \rangle$ for $1 \leq i \leq k$. We then show that $T_1, T_2, \ldots, T_k$ are completely independent spanning trees in $T^* = T_1 \cup T_2 \cup \cdots \cup T_k$ such that each T_i can be embedded in one page under σ, which implies that the augmented graph $T^* \supseteq T$ is a minimum k-arbor-connected graph with pagenumber k.

The graph $\langle E_i \rangle$ is a disjoint union of stars whose central vertices are in W_{i-1}. The augmented graph $\langle E_i \cup A_i \rangle$ is a disjoint union of $|V_{i-1}|$ trees, each of which is obtained from the stars in $\langle E_i \rangle$ by joining each vertex in W_{i-1} and all its ℓ-descendants for $2 \leq \ell \leq k$. Thus, $V(\langle E_i \cup A_i \rangle) = V(T) - \cup_{0 \leq j < i-1} V_j$. The $|V_{i-1}|$ subtrees are connected by the edges uv for $u \lessdot_{\sigma, V_{i-1}} v$ in B_i, and moreover all the vertices in $\cup_{0 \leq j < i-1} V_j$ are joined to a vertex in V_{i-1} by other edges in $B_i \cup B'_i$. Therefore, $\langle E_i \cup A_i \cup B_i \cup B'_i \rangle$ is a tree with vertex set $V(T)$. Note that any edge in $B_i \cup B'_i$ joins a vertex w in $\cup_{0 \leq j < i-1} V_j$ and a vertex in V_{i-1} which is not a descendant of w. In each T_i, every vertex in $V(T) - W_{i-1}$ is directly joined to a vertex in W_{i-1} which means that every vertex in $V(T) - W_{i-1}$ is a leaf of T_i and $V_I(T_i) \subseteq W_{i-1}$. Since $W_i \cap W_j = \emptyset$ for any $0 \leq i < j < k$, $V_I(T_i) \cap V_I(T_j) = \emptyset$ for any $1 \leq i < j \leq k$. Now assume that $e = uv \in E(T_i) \cap E(T_j)$ for some $i < j$. Then, e is incident to a vertex in W_{i-1} and a vertex in W_{j-1}. If $uv \in B_i \cup B'_i$, then $u \in V_{i-1}$ and v must be in V_ℓ where $0 \leq \ell < i$ which is a contradiction. Thus, $uv \in A_i$ such that $u \in V_{kt+i-1}$, $v \in V_{kt+j-1}$ for some $t \geq 0$. This means that u is an ancestor of v. However, no ancestor of v is joined to v as a leaf of T_j. Therefore, $E(T_i) \cap E(T_j) = \emptyset$ for any $1 \leq i < j \leq k$. Consequently, $T_1, T_2, \ldots, T_k$ are completely independent spanning trees.

The graph $\langle E_i \cup A_i \rangle$ is a disjoint union of $|V_{i-1}|$ trees $S_1, S_2, \ldots, S_{|V_{i-1}|}$ such that the vertex set of each S_i corresponds to the vertex set of a subtree rooted at a vertex in V_{i-1}. From a property of a depth-first-search, for any subtree, the vertices in the subtree are consecutive in $V(T)$ under σ. Thus, it can be inductively shown (on the height) that S_i can be embedded in one page under σ. All the vertices in $\cup_{0\leq j<i-1}V_j$ are isolated in $\langle E_i \cup A_i\rangle$ such that no vertex in $\cup_{0\leq j<i-1}V_j$ is placed between vertices of any subtree S_i. Hence, no crossing of edges is produced by adding the edges in $B_i \cup B'_i$ to $\langle E_i \cup A_i\rangle$. Therefore, each T_i can be embedded in one page under the same vertex-ordering σ.

The vertices x and y can be computed in linear time by applying a breadth-first-search twice. The vertex ordering σ follows from σ^+ which is obtained by applying a depth-first-search to T^+ from z. In the depth-first-search, $p_1(v)$ and $\sigma^{-1}(\max_{u'\in V(T_v)}\sigma(u'))$ can also be found for each vertex v. By a depth-first-search, V_{i-1}, W_{i-1}, the ordering relation $\lessdot_{\sigma,V_{i-1}}$, $\sigma^{-1}(\min_{u'\in V_{i-1}}\sigma(u'))$, and $\sigma^{-1}(\max_{u'\in V_{i-1}}\sigma(u'))$ can be computed in $O(n)$ time. Here, E_i and A_i can be rewritten by

- $E_i = \{p_1(v)v \mid v \notin \{x,y\}, v \in W_{i \bmod k}\}$,
- $A_i = \{p_j(v)v \mid v \in V(T),\ p_j(v) \in W_{i-1},\ 2 \leq j \leq k\}$.

For each vertex v and each $1 < j \leq k$, $p_j(v)$ can be computed in $O(k)$ time. Therefore, the sets E_i, A_i, B_i, and B'_i for $1 \leq i \leq k$ can be computed in $O(nk)$ time. ◀

When $k = 2$, a star is the only tree to which Theorem 3.1 cannot be applied. However, a star can be easily augmented to a minimum 2-arbor-connected graph with pagenumber 2 as follows. Let S_n be a star with vertex set $\{v_1, v_2, \ldots, v_n\}$ such that v_1 is the central vertex of S_n. We augment S_n to a wheel graph W_n by adding the edges in $\{v_iv_{i+1} \mid 2 \leq i < n\} \cup \{v_2v_n\}$. Let $E_1 = \{v_1v_i \mid 2 \leq i < n\} \cup \{v_2v_n\}$ and $E_2 = \{v_iv_{i+1} \mid 2 \leq i < n\} \cup \{v_1v_n\}$. Then, $\langle E_1\rangle$ and $\langle E_2\rangle$ are edge-disjoint spanning trees such that $V_I(\langle E_1\rangle) = \{v_1, v_2\}$ and $V_I(\langle E_2\rangle) = \{v_3, \ldots, v_n\}$. Thus, $\langle E_1\rangle$ and $\langle E_2\rangle$ are completely independent spanning trees, and therefore W_n is a minimum 2-arbor-connected graph. Employing the vertex-ordering σ defined as $v_1 <_\sigma v_n <_\sigma v_2 <_\sigma \ldots <_\sigma v_{n-1}$, each $\langle E_i\rangle$ can be embedded in one page under σ. Thus, we have the following corollary.

▶ **Corollary 3.2.** *Any tree can be augmented to a minimum* 2*-arbor-connected graph with pagenumber* 2 *in linear time.*

It follows from Corollary 3.2 that any tree can be augmented to a minimum planar 2-arbor-connected graph in linear time, since a graph with pagenumber 2 is planar.

We here remark that in the proof of Theorem 3.1 other constructions can be employed if we do not insist on the upper bound on k. Select a path P with $|V(P)| \geq 3$ and consider the $|V(P)|$ subtrees each of which is rooted at a vertex in P (instead of two subtrees T_x and T_y). Then, we can construct a minimum k-arbor-connected graph where k is at most the maximum j such that there exist two vertices in P both of which have a $(j-1)$-descendant. In fact, we employ such a construction to prove Theorem 3.5.

Given a graph G with a vertex-ordering σ of a t-page book-embedding of G, let P be the path with $V(P) = V(G)$ and $E(P) = \{\sigma^{-1}(i)\sigma^{-1}(i+1) \mid 1 \leq i < n, i \neq \lfloor \frac{n}{2} \rfloor\} \cup \{\sigma^{-1}(1)\sigma^{-1}(\lfloor \frac{n}{2} \rfloor + 1)\}$. According to the proof of Theorem 3.1, we augment P to P^*. Then, $G \cup P^*$ is a k-arbor-connected graph with pagenumber at most $t + k$. From this observation, we have the following existential result.

▶ **Corollary 3.3.** *For any graph G with n vertices and for any $2 \leq k \leq \frac{n}{2}$, there exists a k-arbor-connected graph $G^* \supseteq G$ with* $\mathrm{pn}(G^*) \leq \mathrm{pn}(G) + k$.

Next, we extend Theorem 3.1 to the class of cacti.

▶ **Theorem 3.4.** *Any cactus G can be augmented to a minimum k-arbor-connected graph with pagenumber k for any $\lfloor \frac{\ell_G}{2} \rfloor + 1 \leq k \leq \mathrm{rad}(G)$ in $O(nk)$ time, where ℓ_G is the maximum length of a cycle in G.*

Proof. Let x be a central vertex of G and y a vertex adjacent to x such that y is on a path between x and a vertex v with $d_G(x, v) = \mathrm{rad}(G)$. Let G^+ be the graph obtained from $G - xy$ by adding a new vertex z with new edges xz and yz. Besides, let T^+ be a breadth-first-search tree of G^+ from z. For each cycle C in G, there is exactly one cycle edge in $E(C) \cap (E(G^+) - E(T^+))$ and we denote by $f(C)$ the cycle edge. Now let $V_i = D_{i+1}(z)$ for $0 \leq i < \mathrm{rad}(G)$ and $W_\ell = \bigcup_{i \bmod k=\ell} V_i$ for each $0 \leq \ell < k$. Consider a depth-first-search ordering σ^+ of the tree T^+ from z such that for any $f(C) = a_C b_C$ with $a_C <_{\sigma^+} b_C$,

- if $a_C \in V_i$, then $b_C \in V_i \cup V_{i-1}$,
- if $a_C <_{\sigma^+} v <_{\sigma^+} b_C$, then v is either a descendant of a_C or an ancestor of b_C.

Define σ as $\sigma(v) = \sigma^+(v) - 1$ for any $v \in V(G)$. Similarly to the proof of Theorem 3.1, we define E_i, A_i, B_i, B'_i and then let $T_i = \langle E_i \cup A_i \cup B_i \cup B'_i \rangle$ for $1 \leq i \leq k$. Consider a cycle edge $f(C) = a_C b_C$ with $a_C <_\sigma b_C$. If $a_C \in V_i$ where $i < k$, then the cycle edge is used in T_i since $f(C) \in \cup_{1 \leq i \leq k} B_i$ from the properties of σ^+. Note that if xy is on a cycle C', then $f(C') \in \cup_{1 \leq i \leq k} B_i$. Suppose that $a_C \in V_{kt+i}$ where $t \geq 1$ and $0 \leq i < k$. Let r_C be the k-ancestor of a_C. Since $k \geq \lfloor \frac{\ell_G}{2} \rfloor + 1$, the subtree rooted at r_C contains the vertex $\mathrm{lca}(a_C, b_C)$. Let $M(C) = \{w \mid \sigma^{-1}(\max_{v \in V(T_{a_C})} \sigma(v)) <_\sigma w \leq_\sigma b_C\}$. Note that $\{r_C w \mid w \in M(C)\} \subseteq A_i$. Replace $E(T_i)$ with $(E(T_i) - \{r_C w \mid w \in M(C)\}) \cup \{a_C w \mid w \in M(C)\}$. Let $T'_1, T'_2, \ldots, T'_k$ be the spanning trees obtained by doing the modification for each cycle edge $f(C)$. Since any edge in $\{a_C w \mid w \in M(C)\}$ is not used in $T_1 \cup T_2 \cup \cdots \cup T_k$, the resultant spanning trees are completely independent spanning trees such that their union contains G. Besides, from the second property of σ^+, each T'_i can be embedded in one page under σ.

It has been shown in [11] that the center of a cactus can be found in linear time. Thus, x (and also y) can be found in linear time. By applying a breadth-first-search to G^+ from z, we can find $f(C)$ for each cycle C and label the end-vertices so that $d_{G^+}(z, a_C) \geq d_{G^+}(z, b_C)$. Besides, we can find $\mathrm{lca}(a_C, b_C)$ and recognize all the edges of C in $O(k)$ time. Let a'_C (resp., b'_C) be the vertex adjacent to $\mathrm{lca}(a_C, b_C)$ on the path from $\mathrm{lca}(a_C, b_C)$ to a_C (resp., b_C). We then apply a depth-first-search in which for each cycle C, each edge $p_1(v)v$ on the path from a'_C to a_C is traversed as the last edge in $\{p_1(v)w \mid w \in D_1(p_1(v))\}$ for the search of $T_{a'_C}$ and just after the search of $T_{a'_C}$, the traversal proceeds through the path from b'_C to b_C. This depth-first-search generates σ^+ satisfying the above two properties in $O(n)$ time. For each cycle edge $f(C)$, the corresponding modification can be done in $O(k)$ time and the number of cycle edges is at most $\lfloor \frac{n-1}{2} \rfloor$. Therefore, we can obtain a minimum k-arbor-connected graph containing G with pagenumber k in $O(nk)$ time. ◀

Although Theorem 3.4 cannot be applied to cycles, we can show the following result.

▶ **Theorem 3.5.** *Any cycle C with n vertices can be augmented to a minimum k-arbor-connected graph with pagenumber k for any $2 \leq k \leq \frac{n}{2}$.*

Proof. Let T be the path obtained from C by deleting one edge ab of C. Suppose that n is even. Let x and y be the central vertices of T. Construct a minimum k-arbor-connected graph T^* according to the construction in the proof of Theorem 3.1. Let $q = \frac{n-2}{2} \bmod k$. Let $V_q = \{a', b'\}$ such that $d_T(a, a') = d_T(b, b') < \frac{n}{2}$. If $q \neq 0$, then by replacing the edges in $B_{q+1} \cup B'_{q+1}$ with the edges in $\{p_j(b')a \mid 1 \leq j \leq q\} \cup \{p_j(a')b \mid 1 \leq j \leq q\} \cup \{ab\}$ in T_{q+1}, we obtain a desired graph. Suppose that $q = 0$ and $x'x, xy, yy' \in E(T)$. Define T_4^+ as the tree obtained from T by deleting the edges $x'x, xy, yy'$ and adding the new vertex z with the edges zx', zx, zy, zy'. Based on T_4^+ instead of T^+ in the proof of Theorem 3.1, we construct $T_1, T_2, \ldots, T_k$ under the condition $x' <_\sigma x <_\sigma y <_\sigma y'$. Note that in this construction, $V_0 = \{x', x, y, y'\}$, $B_1 = \{x'x, xy, yy'\}$ and $a, b \in W_{k-1}$. By modifying T_k in a similar fashion for T_{q+1} in the case that $q \neq 0$, we have a desired graph. Suppose that n is odd and x is the center of T such that $x_1x_2, x_2x, xx_3, x_3x_4 \in E(T)$. Let $r = \frac{n-3}{2} \bmod k$. Define T_3^+ (resp., T_5^+) as the tree obtained from T by deleting the edges x_2x, xx_3 (resp., $x_1x_2, x_2x, xx_3, x_3x_4$) and adding the new vertex z with the edges zx_2, zx, zx_3 (resp., $zx_1, zx_2, zx, zx_3, zx_4$). Similarly to the case that n is even and $q = 0$, we have the desired result by considering T_3^+ (resp., T_5^+) and modifying T_{r+1} (resp., T_k) if $r \neq 0$ (resp., $r = 0$). ◀

We finally remark that the constructions shown in Theorem 3.5 can be generalized to a unicyclic graph G for $2 \leq k \leq \frac{|V(C(G))|}{2}$ by additional discussions.

4 Conclusion

We have shown that any tree with n vertices can be augmented in $O(nk)$ time to a minimum k-arbor-connected graph with pagenumber k for any k at most the radius of the tree. Besides, we have extended the result to cacti and presented an augmentation result for cycles.

It would be interesting to consider augmentation problems for a tree to a minimum k-arbor-connected graph while preserving other good geometric properties of a tree.

Acknowledgments. The author is grateful to the anonymous reviewers for their helpful comments.

References

1. J. Balogh and G. Salazar, Book embeddings of regular graphs, *SIAM J. Discrete Math.* 29 (2015) 811–822.
2. F. Bernhart and P.C. Kainen, The book thickness of a graph, *J. Combin. Theory* Ser. B 27 (1979) 320–331.
3. B. Cheng, D. Wang, and J. Fan, Constructing completely independent spanning trees in crossed cubes, *Discrete Appl. Math.* 219 (2017) 100–109.
4. F.R.K. Chung, F.T. Leighton, and A.L. Rosenberg, Embedding graphs in books: a layout problem with application to VLSI design, *SIAM J. Alg. Discr. Meth.* 8 (1987) 33–58.
5. B. Darties, N. Gastineau, and O. Togni, Completely independent spanning trees in some regular graphs, *Discrete Appl. Math.* 217 (2017) 163–174.
6. T. Hasunuma, Completely independent spanning trees in the underlying graph of a line digraph, *Discrete Math.* 234 (2001) 149–157.
7. T. Hasunuma, Completely independent spanning trees in maximal planar graphs, Lecture Notes in Comput. Sci. 2573, pp. 235–245, 2002, Springer.
8. T. Hasunuma, Minimum degree conditions and optimal graphs for completely independent spanning trees, Lecture Notes in Comput. Sci. 9538, pp. 260–273, 2016, Springer.
9. F. Hurtado and C. Tóth, Plane geometric graph augmentation: a generic perspective, In: Pach J. (eds) Thirty Essays on Geometric Graph Theory, Springer, 2013, pp. 327–354.
10. G. Kant and H. L. Bodlaender, Planar graph augmentation problems, Lecture Notes in Comput. Sci., 519, pp. 286-298, 1991, Springer.
11. Y.-F. Lan, Y.-L. Wang, and H. Suzuki, A linear-time algorithm for solving the center problem on weighted cactus graphs, *Inform. Process. Lett.* 71 (1999) 205–212.
12. K-J. Pai and J-M. Chang, Constructing two completely independent spanning trees in hypercube-variant networks, *Theor. Comput. Sci.* 652 (2016) 28–37.
13. F. Péterfalvi, Two counterexamples on completely independent spanning trees, *Discrete Math.* 312 (2012) 808–810.
14. I. Rutter and A. Wolff, Augmenting the connectivity of planar and geometric graphs, *Electronic Notes in Discrete Math.* 31 (2008) 53–56.
15. M. Yannakakis, Embedding planar graphs in four pages, *J. Comput. System Sci.* 38 (1989) 36–67.