Properties of Minimal-Perimeter Polyominoes*

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Abstract

A polyomino is a set of connected squares on a grid. In this work we address the class of polyominoes with minimal perimeter for their area, and show a bijection between minimal-perimeter polyominoes of certain areas.

1 Introduction

A polyomino is an edge-connected set of cells on the square lattice. The area of a polyomino is the number of cells it contains. The problem of counting polyominoes dates back to the 1950s when it was studied in parallel in the fields of combinatorics [8] and statistical physics [6]. Let $A(n)$ denote the number of polyominoes of area n. A general formula for $A(n)$ is still unknown. Klarner [10] showed the existence of the *growth rate* of $A(n)$, denoting it by $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$. The exact value of λ is also unknown yet, and its best estimate, 4.06, is by Jensen [9]. The current best lower and upper bounds on λ are 4.0025 [3] and 4.6496 [11], respectively. Several works provide enumeration by area of special classes of polyominoes, such as column-convex [7], convex [5], and directed [4] polyominoes.

The perimeter of a polyomino P consists of the empty cells adjacent to P. Asinowski et al. [2] showed that a polyomino of area n has a perimeter of size at most $2n+2$, and provided formulae for the numbers of polyominoes with area n and perimeter $2n+2-k$, for some small values of k. In this paper, we shed some light on polyominoes with the *minimum*-size perimeter for their area. Related works are by Altshuler et al. [1] and by Sieben [12], providing a formula for the maximum area of a polyomino with a certain perimeter size. Sieben [12] also gave a formula for the minimum perimeter size of a polyomino of area n. Both works also characterized all polyominoes that have the maximum area for a given perimeter size. In this paper, we study the number of polyominoes which have the minimum perimeter size for their area, and show a bijection between some sets of minimal-perimeter polyominoes.

2 The Problem

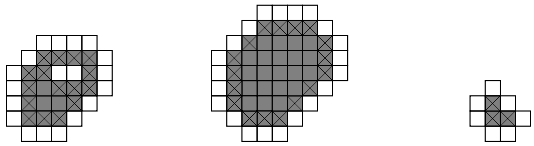
2.1 Definitions

Let Q be a polyomino, and let $\mathcal{P}(Q)$ be the perimeter of Q. Define $\mathcal{B}(Q)$, the *border* of Q, to be the set of cells of Q which have at least one empty neighboring cell. Given a polyomino Q, its *inflated* polyomino, $I(Q)$, is defined as $I(Q) = Q \cup \mathcal{P}(Q)$. Notice that the border of $I(Q)$ is a subset of the perimeter of Q. Analogously, the *deflated* polyomino, $D(Q)$, is defined as $D(Q) = Q \setminus \mathcal{B}(Q)$, which is obtained by "shaving" the outer layer, i.e., the border cells from the polyomino. Notice that the perimeter of $D(Q)$ is a subset of the border of Q. Also note that $D(Q)$ is not necessarily a valid polyomino since the removal of the border of Q may break it into disconnected pieces. Figure 1 demonstrates all the above definitions.

Following the notation of Sieben [12], we denote by $\epsilon(n)$ the minimum size of the perimeter of all polyominoes of area n. Sieben showed that $\epsilon(n) = \left\lceil 2 + \sqrt{8n-4} \right\rceil$. A polyomino Q of area n will be called a minimal-perimeter polyomino if $|\mathcal{P}(Q)| = \epsilon(n)$.

* Work on this paper by both authors has been supported in part by ISF Grant 575/15.

34th European Workshop on Computational Geometry, Berlin, Germany, March 21–23, 2018.



(a) Polyomino Q **(b)** $I(Q)$ **(c)** $D(Q)$

Figure 1 A polyomino Q, its inflated polyomino, and its deflated polyomino. The gray cells are the polyomino cells, while the white cells are the perimeter. Border cells are marked with crosses.

(a) **(b)** **(c)** **(d)** **(w)** **(x)** **(y)** **(z)**

Figure 2 All possible patterns of excess cells. The gray cells are polyomino cells, while the white cells are perimeter cells. Patterns (a–d) exhibit excess border cells and their surrounding perimeter cells, while Patterns (w–z) exhibit excess perimeter cells and their surrounding polyomino cells.

2.2 The Relation between Border, Perimeter, and Excess

In this section we express the size of the perimeter of a polyomino, $|\mathcal{P}(Q)|$, as a function of the border size, $|\mathcal{B}(Q)|$, and the number of excess cells as defined below. The excess of a perimeter cell [2] is defined as the number of polyomino cells that are adjacent to it minus one, and the total excess of a polyomino Q, e_P, is defined as the sum of excess over all the cells of the perimeter of Q. Similarly, the excess of a border cell is defined as the number of perimeter cells adjacent to it minus one, and the border excess, denoted by e_B, is defined as the sum of excess over all the border cells. Let $\pi = |\mathcal{P}(Q)|$ and $\beta = |\mathcal{B}(Q)|$.

▶ **Observation 2.1.** The following holds for any polyomino: $\pi + e_P = \beta + e_B$. Equivalently,

$$\pi = \beta + e_B - e_P. \tag{1}$$

Equation (1) holds since both $\pi + e_P$ and $\beta + e_B$ are equal to the total length of the polygons forming the boundary of the polyomino. This quantity can be calculated either by summing up over the perimeter cells, where each cell contributes 1 plus its excess for a total of $\pi + e_P$, or by summing up over the border cells for a total of $\beta + e_B$. Figure 2 shows all possible patterns of border and perimeter excess cells, while Figure 3 shows a sample polyomino with some cells tagged with the corresponding patterns.

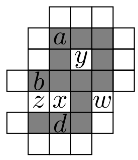


Figure 3 A sample polyomino with marked patterns.

Let #☐ be the number of excess cells of a certain type in a polyomino as classified in the figure, where '☐' is one of the symbols a–d or w–z, as in Figure 2. Counting e_P and e_B as functions of the different patterns of excess cells, we see that $e_B = \#a + 2\#b + 3\#c + \#d$ and $e_P = \#w + 2\#x + 3\#y + \#z$. Substituting e_B and e_P in Equation (1), we obtain

$$\pi = \beta + \#a + 2\#b + 3\#c + \#d - \#w - 2\#x - 3\#y - \#z.$$

Since Pattern (c) is a singleton cell, we can ignore it in the general formula. Thus, we have

$$\pi = \beta + \#a + 2\#b + \#d - \#w - 2\#x - 3\#y - \#z.$$

2.3 Properties of Minimal-Perimeter Polyominoes

▶ **Lemma 2.2.** *Any minimal-perimeter polyomino is simply connected (that is, it does not contain holes).*

(a) Q (b) Q' (c) Q (d) Q'

Figure 4 Examples for the first and second parts of the proof of Theorem 2.4.

Proof. The sequence $\epsilon(n)$ is monotone increasing in the wide sense[1] [12]. Assume that there exists a minimal-perimeter polyomino Q with a hole. Consider the polyomino Q' that is obtained by filling this hole. The area of Q' is clearly larger than the area of Q, and its perimeter size is smaller since we eliminated the perimeter cells inside the hole and did not introduce new perimeter cells. This is a contradiction to $\epsilon(n)$ being monotone increasing. ◀

▶ **Lemma 2.3.** *For a simply connected polyomino, we have* $\#a + 2\#b - \#w - 2\#x = 4$.

Proof. The boundary of a polyomino without holes is a simple polygon, thus, the sum of its internal angles is $(180(v-2))°$, where v is the complexity of the polygon. Notice that Pattern (a) (resp., (b)) adds one (resp., two) 90°-vertex to the polygon. Similarly, Pattern (w) (resp. (x)) adds one (resp., two) 270°-vertex. All other patterns do not involve vertices. Let $L = \#a + 2\#b$ and $R = \#w + 2\#x$. Then, the sum of angles of the boundary polygon implies that $L \cdot 90° + R \cdot 270° = (L + R - 2) \cdot 180°$, that is, $L - R = 4$. The claim follows. ◀

▶ **Theorem 2.4.** *(Stepping Theorem) For a minimal-perimeter polyomino (except the singleton cell), we have that* $\pi = \beta + 4$.

Proof. Lemma 2.3 tells us that $\pi = \beta+4+\#d-\#z$. We will show that any minimal-perimeter polyomino contains neither Pattern (d) nor Pattern (z).

Let Q be a minimal-perimeter polyomino. For the sake of contradiction, assume first that there is a cell $f \in \mathcal{P}(Q)$ as part of Pattern (z). Assume w.l.o.g. that the two adjacent polyomino cells are to the left and to the right of f. These two cells must be connected, thus, the area below (or above) f must be bounded by polyomino cells. Let, then, Q' be the polyomino with the area below f, and the cell f itself, filled with polyomino cells. The cell directly above f becomes a perimeter cell, the cell f ceases to be a perimeter cell, and at least one perimeter cell in the area filled below f is eliminated, thus, $|\mathcal{P}(Q')| < |\mathcal{P}(Q)|$ and $|Q'| > |Q|$, which is a contradiction to the sequence $\epsilon(n)$ being increasing. Thus, Q does not contain perimeter cells that fit Pattern (z). Figures 4(a,b) demonstrate this argument.

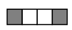
Now assume for contradiction that Q contains a cell f, forming Pattern (d). Let Q' be the polyomino obtained from Q by removing f and then "pushing" together the two cells adjacent to f. This is always possible since Q is of minimal perimeter, hence, by Lemma 2.2, it is simply connected, and thus, removing f breaks Q into two separate polyominoes. Any two separated polyominoes can be shifted by one cell without colliding, thus, the transformation described above is valid. The area of Q' is one less than the area of Q, and the perimeter of Q' is smaller by at least two than the perimeter of Q, since the perimeter cells below and above f cease to be part of the perimeter, and connecting the two parts does not create new perimeter cells. From the formula of $\epsilon(n)$ we know that $\epsilon(n+1) - \epsilon(n) \leq 1$ for $n \geq 2$, but $|Q| - |Q'| = 1$ and $|\mathcal{P}(Q)| - |\mathcal{P}(Q')| = 2$, hence, Q is not a minimal-perimeter polyomino, which contradicts our assumption. Thus, there are no cells in Q that fit Pattern (d). Figures 4(c,d) demonstrate this argument. This completes the proof. ◀

[1] In the sequel we simple say "monotone increasing."

2.4 Inflating a Minimal-Perimeter Polyomino

In this section we reach our main results.

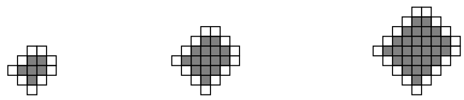
▶ **Lemma 2.5.** *If Q is a minimal-perimeter polyomino, then $|\mathcal{P}(I(Q))| \leq |\mathcal{P}(Q)| + 4$.*

Proof. Since Q is a minimal-perimeter polyomino, we know by Lemma 2.2 that $I(Q)$ is simply connected. For a hole to be formed in $I(Q)$, the original polyomino Q must have either Pattern (z) (two cells separated by a single perimeter cell), or two cells separated by *two* perimeter cells, as in . The former case (Pattern (z)) is not possible, as is shown in the proof of Theorem 2.4. We show, using the same technique, that the latter case is also impossible.

Since $I(Q)$ is simply connected, we have, by Lemma 2.3, that $|\mathcal{P}(I(Q))| = |\mathcal{B}(I(Q))| + 4 + \#d - \#z$. Since $|\mathcal{B}(I(Q))| \leq |\mathcal{P}(Q)|$, all that remains to show is that Pattern (d) does not occur in $I(Q)$. Assume to the contrary that there is a cell f forming Pattern (d) in $I(Q)$. Since $I(Q)$ is simply connected, removing f will break it into exactly two pieces, denoted by Q_1 and Q_2. Both Q_1 and Q_2 must contain cells of the original Q since any cell in $I(Q)$ either belongs to Q or is adjacent to a cell of Q. However, this implies that Q is not connected, which is a contradiction. Hence, Q cannot contain a pattern of type (d), as required. ◀

▶ **Theorem 2.6.** *(Inheritance Theorem) If Q is a minimal-perimeter polyomino, then $I(Q)$ is a minimal-perimeter polyomino as well.*

Proof. Let Q be a minimal-perimeter polyomino. Assume to the contrary that $I(Q)$ is not a minimal-perimeter polyomino, i.e., there exists a polyomino Q' with the same area as $I(Q)$, such that $|\mathcal{P}(Q')| < |\mathcal{P}(I(Q))|$. From Lemma 2.5 we know that $|\mathcal{P}(I(Q))| \leq |\mathcal{P}(Q)| + 4$, thus, the perimeter of Q' is at most $|\mathcal{P}(Q)| + 3$, and since Q' is a minimal-perimeter polyomino, we know by Theorem 2.4 that the size of its border is at most $|\mathcal{P}(Q)| - 1$. Consider now $D(Q')$. The area of Q' is $|Q| + |\mathcal{P}(Q)|$, thus, the size of $D(Q')$ is at least $|Q| + 1$, and its perimeter size is at most $\epsilon(n) - 1$ (since the perimeter of $D(Q')$ is a subset of the border of Q'). This is a contradiction to the sequence $\epsilon(n)$ being monotone increasing. Hence, Q' cannot exist, and $I(Q)$ is a minimal-perimeter polyomino. Figure 5 demonstrates this theorem. It shows a minimal-perimeter polyomino Q of area 6 and the two minimal-perimeter polyominoes of areas 15 and 28 obtained by inflating Q twice. ◀



(a) $|Q| = 6$ **(b)** $|I(Q)| = 15$ **(c)** $|I(I(Q))| = 28$

Figure 5 A demonstration of Theorem 2.6.

▶ **Corollary 2.7.** *The minimum perimeter size of a polyomino of area $n + k\epsilon(n) + 2k(k-1)$ (for $n \neq 1$ and any $k \in \mathbb{N}$) is $\epsilon(n) + 4k$.*

Proof. Inflating a minimal-perimeter polyomino of size n increases its area by $\epsilon(n)$. The border size of the inflated polyomino is $\epsilon(n)$, thus, by Theorem 2.4, the new perimeter size is $\epsilon(n) + 4$. By induction, after the kth inflation, the perimeter size is $\epsilon(n) + 4k$ and the increase in the area is $\epsilon(n) + 4(k-1)$. Summing up the increase in area, we obtain $\sum_{i=1}^{k}(\epsilon(n) + 4(i-1)) = k\epsilon(n) + 2k(k-1)$, implying the claim. ◀

▶ **Lemma 2.8.** *Let Q be a minimal-perimeter polyomino of area $n + \epsilon(n)$ (for $n \geq 3$). Then, $D(Q)$ is a valid (connected) polyomino.*

Proof. Assume to the contrary that $D(Q)$ is not connected and that it is composed of at least two parts. Assume first that $D(Q)$ is composed of exactly two parts, Q_1 and Q_2.

Define the *joint perimeter* of the two parts, $\mathcal{P}(Q_1, Q_2)$, to be $\mathcal{P}(Q_1) \cup \mathcal{P}(Q_2)$. Since Q is a minimal-perimeter polyomino of area $n + \epsilon(n)$, we know that its perimeter size is $\epsilon(n) + 4$ and its border size is $\epsilon(n)$, by Corollary 2.7 and Theorem 2.4, respectively. Thus, the size of $D(Q)$ is exactly n regardless of whether or not $D(Q)$ is connected. Since Q_1 and Q_2 are the result of deflating Q, the polyomino Q must have an (either horizontal, vertical, or diagonal) "bridge" of border cells which disappeared in the deflation. The width of the bridge is at most 2, thus, $|\mathcal{P}(Q_1) \cap \mathcal{P}(Q_2)| \le 2$. Hence, $|\mathcal{P}(Q_1)| + |\mathcal{P}(Q_2)| - 2 \le |\mathcal{P}(Q_1, Q_2)|$. Since $\mathcal{P}(Q_1, Q_2)$ is a subset of $\mathcal{B}(Q)$, we have that $|\mathcal{P}(Q_1, Q_2)| \le \epsilon(n)$. Therefore,

$$\epsilon(|Q_1|) + \epsilon(|Q_2|) - 2 \le \epsilon(n). \quad (2)$$

Recall that $|Q_1| + |Q_2| = n$. It is easy to observe that $\epsilon(|Q_1|) + \epsilon(|Q_2|)$ is minimized when $|Q_1| = 1$ and $|Q_2| = n - 1$ (or vice versa). Had the function $\epsilon(n)$ (shown in Figure 6) been $2 + \sqrt{8n - 4}$ (without rounding up), this would be obvious. But since $\epsilon(n) = \lceil 2 + \sqrt{8n - 4} \rceil$, it is a step function (with an infinite number of intervals), where the gap between all successive steps is exactly 1, except the gap between the two leftmost steps which is 2. This guarantees that despite the rounding, the minimum of $\epsilon(|Q_1|) + \epsilon(|Q_2|)$ occurs as claimed. Substituting this into Equation (2), and using the fact that $\epsilon(1) = 4$, we see that $\epsilon(n-1) + 2 \le \epsilon(n)$. However, we know [12] that $\epsilon(n) - \epsilon(n-1) \le 1$ for $n \ge 3$, which is a contradiction. Thus, $D(Q)$ cannot split into two parts unless it splits into two singleton cells, which is indeed the case for a minimal-perimeter polyomino of size 8.

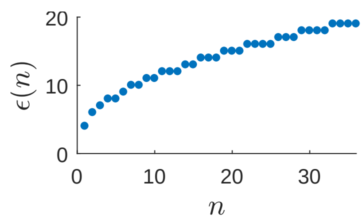


Figure 6 Values of $\epsilon(n)$.

The same method can be used to show that $D(Q)$ cannot be composed of more then two parts. Note that this proof does not hold for polyominoes of area which is not of the form $n + \epsilon(n)$, but it suffices for the proof of Theorem 2.10 below. ◀

▶ **Lemma 2.9.** *Let Q_1, Q_2 be two different minimal-perimeter polyominoes. Then, regardless of whether or not Q_1, Q_2 have the same area, $I(Q_1)$ and $I(Q_2)$ are different as well.*

Proof. Assume to the contrary that $Q = I(Q_1) = I(Q_2)$. By definition, this means that $Q = Q_1 \cup \mathcal{P}(Q_1) = Q_2 \cup \mathcal{P}(Q_2)$. Furthermore, since $Q_1 \ne Q_2$, and since a cell can belong to either a polyomino or to its perimeter, but not to both, it must be that $\mathcal{P}(Q_1) \ne \mathcal{P}(Q_2)$. The border of Q is a subset of both $\mathcal{P}(Q_1)$ and $\mathcal{P}(Q_2)$, that is, $\mathcal{B}(Q) \subset \mathcal{P}(Q_1) \cap \mathcal{P}(Q_2)$. Since $\mathcal{P}(Q_1) \ne \mathcal{P}(Q_2)$, we have that either $|\mathcal{B}(Q)| < |\mathcal{P}(Q_1)|$ or $|\mathcal{B}(Q)| < |\mathcal{P}(Q_2)|$; assume w.l.o.g. the former case. Now consider the polyomino $D(Q)$. Its area is $|Q| - |\mathcal{B}(Q)|$. The area of Q is $|Q_1| + |\mathcal{P}(Q_1)|$, thus, $|D(Q)| > |Q_1|$, and since the perimeter of $D(Q)$ is a subset of the border of Q, we conclude that $|\mathcal{P}(D(Q))| < |\mathcal{P}(Q_1)|$. However, Q_1 is a minimal-perimeter polyomino, which is a contradiction to $\epsilon(n)$ being monotone increasing. ◀

▶ **Theorem 2.10.** *(Chain Theorem) Let M_n be the set of minimal-perimeter polyominoes of area n. Then, for $n \ge 3$, we have that $|M_n| = \left|M_{n+\epsilon(n)}\right|$.*

Proof. By Theorem 2.6, if $Q \in M_n$, then $I(Q) \in M_{n+\epsilon(n)}$, and hence, by Lemma 2.9, we have that $|M_n| \le \left|M_{n+\epsilon(n)}\right|$. Let us now show the opposite relation, namely, that $|M_n| \ge \left|M_{n+\epsilon(n)}\right|$. The combination of the two relations will imply the claim.

Let $I(M_n) = \{I(Q) \mid Q \in M_n\}$. For $Q \in M_{n+\epsilon(n)}$, our goal is to show that $Q \in I(M_n)$. Since $Q \in M_{n+\epsilon(n)}$, we have by Corollary 2.7 that $|\mathcal{P}(Q)| = \epsilon(n) + 4$. Moreover, by Theorem 2.4, we have that $|\mathcal{B}(Q)| = \epsilon(n)$, thus, $|D(Q)| = n$ and $|\mathcal{P}(D(Q))| \ge \epsilon(n)$. Since the perimeter of $D(Q)$ is a subset of the border of Q, and $|\mathcal{B}(Q)| = \epsilon(n)$, we conclude that the perimeter of $D(Q)$ and the border of Q are the same set of cells. Thus, $I(D(Q)) = Q$. Since

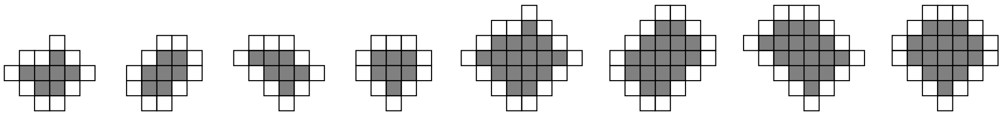


Figure 7 A demonstration of Theorem 2.10.

$|\mathcal{P}(D(Q))| = \epsilon(n)$, we have that $D(Q)$ is a minimal-perimeter polyomino, thus, $Q \in I(M_n)$ as required. Hence, $M_{n+\epsilon(n)} \subseteq I(M_n)$, implying that $\left|M_{n+\epsilon(n)}\right| \leq |I(M_n)| = |M_n|$.

Figure 7 shows, for example, all minimal-perimeter polyominoes of area 7. When they are inflated, they become the entire set of minimal-perimeter polyominoes of area 17. ◀

▶ **Corollary 2.11.** *For $n \geq 3$ and any $k \in \mathbb{N}$, we have that $|M_n| = \left|M_{n+k\epsilon(n)+2k(k-1)}\right|$.*

Proof. The claim follows from applying Theorem 2.10 repeatedly on M_n. ◀

3 Future work

We have shown that inflating a set of minimal-perimeter polyominoes of a certain area creates a new set, of the same cardinality, of minimal-perimeter polyominoes of some other area. This creates chains of sets of minimal-perimeter polyominoes of the same area. In the future we would like to characterize the roots of these chains and to determine how many minimal-perimeter polyominoes the sets of each chain contains.

References

1. Y. Altshuler, V. Yanovsky, D. Vainsencher, I.A. Wagner, and A.M. Bruckstein. On minimal perimeter polyminoes. In *DGCI*, pages 17–28. Springer, 2006.
2. A. Asinowski, G. Barequet, and Y. Zheng. Enumerating polyominoes with fixed perimeter defect. In *Proc. 9th European Conf. on Combinatorics, Graph Theory, and Applications*, volume 61, pages 61–67, Vienna, Austria, August 2017. Elsevier.
3. G. Barequet, G. Rote, and M. Shalah. $\lambda > 4$: An improved lower bound on the growth constant of polyominoes. *Comm. of the ACM*, 59(7):88–95, 2016.
4. M. Bousquet-Mélou. New enumerative results on two-dimensional directed animals. *Discrete Mathematics*, 180(1-3):73–106, 1998.
5. M. Bousquet-Mélou and J.-M. Fédou. The generating function of convex polyominoes: The resolution of a q-differential system. *Discrete Mathematics*, 137(1-3):53–75, 1995.
6. S.R. Broadbent and J.M. Hammersley. Percolation processes: I. Crystals and Mazes. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 53, pages 629–641. Cambridge University Press, 1957.
7. M.-P. Delest. Generating functions for column-convex polyominoes. *J. of Combinatorial Theory, Series A*, 48(1):12–31, 1988.
8. S.W. Golomb. Checker boards and polyominoes. *The American Mathematical Monthly*, 61(10):675–682, 1954.
9. I. Jensen and A.J. Guttmann. Statistics of lattice animals (polyominoes) and polygons. *J. of Physics A: Mathematical and General*, 33(29):L257, 2000.
10. D.A. Klarner. Cell growth problems. *Canadian J. of Mathematics*, 19:851–863, 1967.
11. D.A. Klarner and R.L. Rivest. A procedure for improving the upper bound for the number of n-ominoes. *Canadian J. of Mathematics*, 25(3):585–602, 1973.
12. N. Sieben. Polyominoes with minimum site-perimeter and full set achievement games. *European J. of Combinatorics*, 29(1):108–117, 2008.