Drawing Connected Planar Clustered Graphs on Disk Arrangements *†

Tamara Mchedlidze[1], Marcel Radermacher[1], Ignaz Rutter[2], and Nina Zimbel[1]

1 **Department of Computer Science, Karlsruhe Institute of Technology, Germany**
`mched@iti.uka.de`, `radermacher@kit.edu`

2 **Department of Mathematics and Computer Science, TU Eindhoven, The Netherlands,** `i.rutter@tue.nl`

Abstract

Let $G = (V, E)$ be a planar graph and let $\mathcal{V}$ be a partition of V whose *clusters*, i.e., the graphs induced by the vertex sets in $\mathcal{V}$, are connected. Let $\mathcal{D}_\mathcal{C}$ be an arrangement of disks with a bijection between the disks and the clusters. Akitaya et al. [1] give an algorithm to test whether $(G, \mathcal{V})$ can be embedded onto $\mathcal{D}_\mathcal{C}$ with the additional constraint that edges are routed through an additional set of pipes between the disks. Based on such an embedding, we prove that every clustered graph with connected clusters and every disk-arrangement with non-overlapping disks has a planar straight-line drawing where every vertex is embedded in the disk corresponding to its cluster. This result can be seen as an extension of the result by Alam et al. [2] who solely consider biconnected clusters.

1 Introduction

In this paper, we study the problem of drawing a large plane clustered graph G on a prescribed disk arrangement $\mathcal{D}_\mathcal{C}$. More formally, a *(flat) clustering* of a graph $G = (V, E)$ is a partition $\mathcal{V} = \{V_1, \dots, V_k\}$ of the vertex set V. We refer to the pair $\mathcal{C} = (G, \mathcal{V})$ as a *clustered graph* and the graphs G_i induced by V_i as *clusters*. A *disk arrangement* $\mathcal{D} = \{d_1, \dots, d_k\}$ is a set of pairwise disjoint disks in the plane together with a bijective mapping $\mu(V_i) = d_i$ between the clusters $\mathcal{C}$ and the disks $\mathcal{D}$. We refer to a disk arrangement $\mathcal{D}$ with a bijective mapping μ as a *disk arrangement of* $\mathcal{C}$, denoted by $\mathcal{D}_\mathcal{C}$. A $\mathcal{D}_\mathcal{C}$*-framed drawing of* $\mathcal{C}$ is a planar drawing of a clustered graph $\mathcal{C}$ where each cluster G_i is drawn within its corresponding disk d_i. We study the following problem: given a clustered planar graph G with an embedding ψ and a disk arrangement $\mathcal{D}_\mathcal{C}$ of $\mathcal{C}$, does G admit a $\mathcal{D}_\mathcal{C}$-framed straight-line drawing homeomorphic to ψ?

A *pipe* p_{ij} *of two clusters* V_i, V_j is the *convex hull* of the disks d_i and d_j, i.e., the smallest convex set of points containing d_i and d_j; see Fig. 1. A disk arrangement $\mathcal{D}_\mathcal{C}$ of $\mathcal{C}$ is *planar* if (i) the pairwise intersections of all disks are empty, and (ii) if $(V_i \times V_j) \cap E \neq \emptyset$, then the intersection of p_{ij} with all disks d_k (corresponding to V_k) is empty (i, j, k pairwise distinct) and, (iii) if $(V_i \times V_j) \cap E \neq \emptyset$ and $(V_k \times V_l) \cap E \neq \emptyset$ (i, j, k, l pairwise distinct), then the intersection of the pipes p_{ij} and p_{kl} is empty. A planar disk arrangement can be seen as a thickening of the graph obtained by contracting all clusters in $\mathcal{C}$. An *embedding* ψ of G, i.e., a topological planar drawing of G, is *compatible* with a planar disk arrangement $\mathcal{D}_\mathcal{C}$ if ψ is homeomorphic to a $\mathcal{D}_\mathcal{C}$-framed embedding of $\mathcal{C}$ such that edges of a cluster are routed within the corresponding disks, and edges between distinct clusters are routed through the

* Work was partially supported by grant WA 654/21-1 of the German Research Foundation (DFG).
† This research was funded in part by Humility & Conviction in Public Life, a project of the University Connecticut sponsored by the John Templeton Foundation.

34th European Workshop on Computational Geometry, Berlin, Germany, March 21–23, 2018.

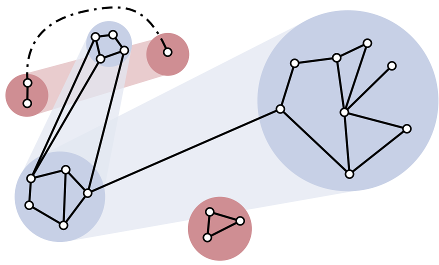


Figure 1 The blue disk arrangement is planar. The red disk arrangement disrupts the planarity of the entire arrangement. The dash dotted edge is not embedded in a pipe, hence the embedding is not compatible with the disk arrangement.

corresponding pipes. Throughout the paper we assume the disk arrangement, provided as part of the input, is planar.

Related Work

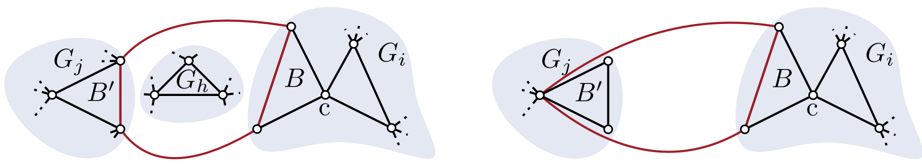
Feng et al. [7] introduced the notion of *clustered graphs* and *c-planarity.* A graph G together with a recursive partitioning of the vertex set is considered to be a clustered graph. An embedding of G is a *c-planar embedding* if (i) each cluster c is drawn within a connected region R_c, (ii) two regions R_c, R_d intersect if and only if the cluster c contains the cluster d or vice versa, and (iii) every edge intersects the boundary of a region at most once. They prove that a c-planar embedding of a connected clustered graph can be computed in $O(n^2)$ time. It is an open question whether it is possible to extend this result to disconnected clustered graphs. Many special cases of this problem have been considered [4].

Concerning drawings of c-planar clustered graphs, Eades et al. [6] prove that every c-planar graph has a c-planar straight-line drawing where each cluster is drawn in a convex region. Angelini et al. [3] strengthen the result of Eades et al. by showing that every c-planar graph has a c-planar straight-line drawing in which every cluster is drawn in an axis-parallel rectangle. The result of Akitaya et al. [1] implies that in $O(n \log n)$ time one can decide whether an abstract graph with a flat clustering has an embedding where each vertex lies in a prescribed topological disk and every edge is routed through a prescribed topological pipe. In general their algorithm decides whether a simplicial map φ of G onto a 2-manifold M is a *weak embedding*, i.e., for every $\epsilon > 0$, φ can be perturbed into an embedding ψ_ϵ with $||\varphi - \psi_\epsilon|| < \epsilon$.

Alam et al. [2] prove that it is $\mathcal{NP}$-hard to decide whether a clustered graph has a c-planar straight-line drawing where every cluster is contained in a prescribed rectangle and edges have to pass through a defined part of the boundary of the rectangle. Further, they prove that all instances with biconnected clusters always admit a solution. Their result implies that graphs of this class have $\mathcal{D}_\mathcal{C}$-framed straight-line drawings.

Contribution

In this paper, we prove that every *connected clustered graph* $(G, \mathcal{V})$, i.e., each cluster G_i is connected, with an embedding ψ compatible with a prescribed planar disk arrangement $\mathcal{D}_\mathcal{C}$, has a $\mathcal{D}_\mathcal{C}$-framed planar straight-line drawing homeomorphic to ψ. Taking the result of Akitaya et al. [1] into account, our result can be used to test whether an abstract clustered graph with connected clusters has a $\mathcal{D}_\mathcal{C}$-framed straight-line drawing. Our result is an extension of the result of Alam et al. [2] from biconnected to connected clusters.



(a) **(b)**

Figure 2 (a) A planar clustered graph $\mathcal{C}$ that is not simple. (b) The block B is leaf block of G_i. The block B' of G_j obstructs B, B' itself is free. The cycles mentioned in the definitions are highlighted in red.

2 Preliminaries

A clustered graph $\mathcal{C} = (G, \mathcal{V})$ is *simple* if for every i, j, there is no cluster $G_h(i, j \neq h)$ embedded in the interior of the subgraph induced by $V_i \cup V_j$; see Fig. 2a. Note that this is a necessary condition in our model, as otherwise the corresponding disk arrangement would not be planar. The set of edges E_i of a cluster G_i are *intra-cluster edges* and the set of edges with endpoints in different clusters *inter-cluster edges*. The vertex u of an inter-cluster edge uv is the *inter-cluster neighbor of* v.

We refer to a maximal biconnected component B of G_i as a *block of* G_i. Removing a *cut vertex* from G_i, splits G_i into two connected components. A block is a *leaf block* if it is incident to at most one cut vertex of G_i; see Fig. 2b. A block B' of a cluster G_j *obstructs* a leaf block of G_i in ψ if there is a cycle C using only vertices of B and at most a single vertex of B' such that B' is in the interior of the graph induced by $C \cup B \cup B'$. A block B that is not obstructed by another block is *free*. We denote the graph after the contraction of a block B by G/B and refer to the resulting vertex b as the *contraction vertex of* G/B. The contraction of a block in a graph with an embedding ψ *induces an embedding* $\psi_{G/B}$ of G/B.

▶ **Lemma 2.1.** *Let $\mathcal{C} = (G, \mathcal{V})$ be a connected simple clustered graph with an embedding ψ that is compatible with a disk arrangement $\mathcal{D}_\mathcal{C}$. Then the embedding induced by the contraction of a free leaf block is compatible with $\mathcal{D}_\mathcal{C}$.*

3 Drawing Planar Clustered Graphs on Disk Arrangements

In this section, we prove that every connected simple clustered graph $\mathcal{C}$ has a $\mathcal{D}_\mathcal{C}$-framed straight-line drawing, see Theorem 3.6. Our proof strategy is as follows. We iteratively contract free leaf blocks B of $\mathcal{C}$ until every cluster contains exactly one vertex, see Lemma 3.1. In this case, the center points of the disks in the disk arrangement $\mathcal{D}_\mathcal{C}$ induce a $\mathcal{D}_\mathcal{C}$-framed straight-line drawing of $\mathcal{C}$. In order to undo a contraction of a free leaf block B, we consider a $\mathcal{D}_\mathcal{C}$-framed straight-line drawing $\Gamma_{\mathcal{C}/B}$ of the contracted graph $\mathcal{C}/B$, see Fig 3b. We start by defining a safe convex polygon σ, that allows us to extend the drawing $\Gamma_{\mathcal{C}/B}$ to a drawing Γ of $\mathcal{C}$, by placing vertices on the boundary of B on the boundary of σ, and the interior vertices of B in the interior of σ. The result of Chambers et al. [5] ensures that the drawing of B, where the vertices on the boundary of B have prescribed placements on the boundary of a convex polygon, is a planar straight-line drawing homeomorphic to the embedding of B. The challenging part is to guarantee that the inter-cluster edges do not intersect with edges of B; see Lemma 3.2 to Lemma 3.5. We first prove that unless the clustered graph is not sufficiently small, there is a free leaf block B.

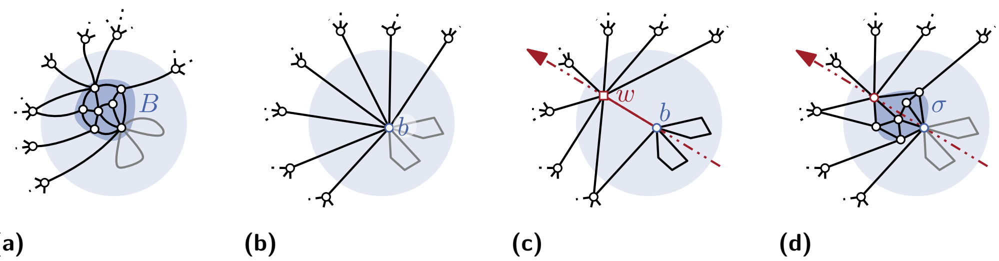


Figure 3 (a) A block B (black) with inter-cluster neighbors outside of the blue disk. (b) A straight-line drawing of the B-contracted graph. (c) A U_b-similar segment $\overline{bw}$ with its supporting line (red). (d) $\mathcal{D}_\mathcal{C}$-framed straight-line drawing with B drawn in the dark blue convex polygon σ.

▶ **Lemma 3.1.** *Every connected simple clustered graph $\mathcal{C} = (G, \mathcal{V})$ has a cluster G_i with a free leaf block or every cluster has exactly one vertex.*

Let B be a free leaf block of a cluster G_i and consider a $\mathcal{D}_\mathcal{C}$-framed straight-line drawing $\Gamma_{\mathcal{C}/B}$ of a B-contracted clustered graph $\mathcal{C}/B$. Observe that we cannot take an arbitrary convex polygon σ to extend the drawing $\Gamma_{\mathcal{C}/B}$ to a drawing Γ, since for this polygon it might not be possible to avoid intersections between inter-cluster edges and edges of B. To avoid these intersections, we construct the polygon σ in two phases. First, we will prove the existence of a special segment s (see Fig 3c), that we will later use to construct two polygons σ_L and σ_R. Then the union of σ_L and σ_R will be the desired polygon σ.

We formalize the concept of a safe point set as follows. Denote by U_b the inter-cluster neighbors of the contraction vertex b and let $L \subseteq U_b$ be a set of vertices that is consecutive in the clockwise order around b. We construct an L-*split drawing* Γ_p from $\Gamma_{\mathcal{C}/B}$ by removing the inter-cluster edges $\{bu \mid u \in L\}$ from $\Gamma_{\mathcal{C}/B}$ and adding a *split vertex* w at position $p \in \mathbb{R}^2$ and connecting w to all vertices in $L \cup \{b\}$ with straight-line edges. We say a set $P \subseteq \mathbb{R}^2$ is L-*similar* if for every point $p \in P$ the L-split drawing Γ_p of $\Gamma_{\mathcal{C}/B}$ is planar, and the contraction of the edge bw induces an embedding homeomorphic to $\Gamma_{\mathcal{C}/B}$.

▶ **Lemma 3.2.** *Let B be a free leaf block of a cluster G_i and let $d_i \in \mathcal{D}_\mathcal{C}$ be the corresponding disk. Let $\Gamma_{\mathcal{C}/B}$ be a $\mathcal{D}_\mathcal{C}$-framed straight-line drawing of $\mathcal{C}/B$. Let b be the contraction vertex of $\mathcal{C}/B$ with inter-cluster neighbors U_b. Then there is a U_b-similar straight-line segment $s \subset d_i$.*

Proof sketch. There is a small disk $\delta \subset d_i$ around b such that moving b within δ preserves the topological properties of b. Let e_l be and e_r be the edges that precede and succeed B, respectively. Then, the two lines containing e_l and e_r divide δ into four regions of which one region R is U_b-similar. Thus, every segment $\overline{ba}$, with $a \in R$, is U_b-similar. ☐

A *supporting line* of a U_b-similar segment $s = \overline{ba}$ is the line that contains s and is directed from b towards a. This line l separates the set U_b into sets L and R, such that the vertices in L are to left of l in the drawing $\Gamma_{\mathcal{C}/B}$, and the vertices in R to the right of l. Depending on the set, we show that there are convex polygons σ_L and σ_R that are monotone with respect to s. For a segment $s = \overline{ba}$, a convex polygon $\langle p_0, p_1, \dots, p_k, p_{k+1}\rangle$, with $p_0 = a$ and $p_{k+1} = b$, is s-*monotone* if the projections of all p_i onto the supporting line of s, lie on s.

▶ **Lemma 3.3.** *Let $\Gamma_{\mathcal{C}/B}$ be a $\mathcal{D}_\mathcal{C}$-framed straight-line drawing of $\mathcal{C}/B$ and let U_b be the inter-cluster neighbors of the contraction vertex b and let $s = \overline{ba}$ be a U_b-similar segment. Let $L \subseteq U_b$ be the set of vertices that are to the left of the supporting-line of s. Then there is a convex s-monotone polygon σ_L contained in $d_i \in \mathcal{D}_\mathcal{C}$ such that the boundary $\mathcal{BD}(\sigma_L)$ of σ_L is L-similar, and for every point p on $\mathcal{BD}(\sigma_L) \setminus s$ and every vertex $u \in L$, the open segment $\overline{pu}$ and σ_L do not intersect.*

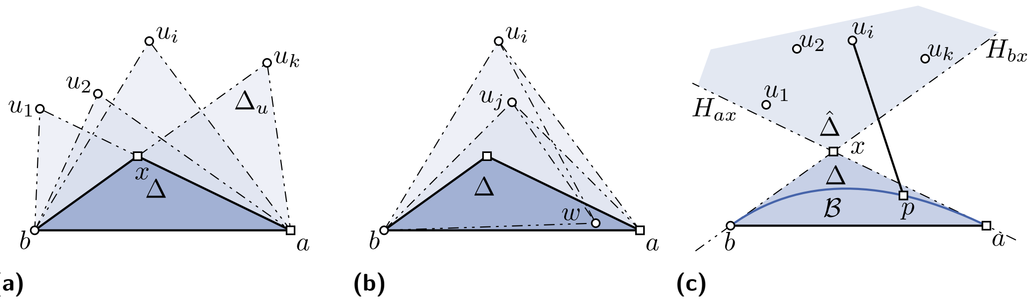


Figure 4 (a) Triangle Δ is the intersection of all triangles Δ_u. (b) Δ is not L-similar. (c) $\mathcal{B}$ is a Bézier-curve within Δ.

Proof sketch. Consider the non-empty set L and the triangle Δ_u with vertices b, u, a for a vertex $u \in L$; see Fig. 4a. Let Δ be the intersection of all triangles Δ_u. Since the segment $s = \overline{ba}$ is U_b-similar and the set L contains all vertices to the left of l, Δ is L-similar. Unfortunately, the triangle $\Delta = (b, x, a)$ is not the desired polygon σ_L, yet. To ensure that the polygon σ_L is s-monotone and entirely contained in d_i, we place the vertex x in the intersection of Δ and d_i, such that the projection of x lies on s. Such a point exists, since s is contained in d_i. Finally, we have to guarantee that for every point p in $\mathcal{BD}(\sigma_L) \setminus s$ and every vertex u in L, the open segment $\overline{pu}$ and σ_L do not intersect. Indeed the Bézier-curve $\mathcal{B}$ with b, x, a as its control points satisfies this property. Hence, the desired polygon σ_L can be constructed by discretizing the curve $\mathcal{B}$. ◻

Observe that this lemma can be restated in terms of the set R right of the supporting line l of s. We then obtain an s-monotone polygon σ_R. Merging the two polygons σ_L and σ_R results in the final polygon σ. Before we are able to actually draw the block B on σ, it is crucial that the notion of vertices to the left and right of a supporting line l transfers to the vertices on the boundary of B. We formalize this with the concept of an *apex vertex of B*. Let $v_0, v_1, \ldots, v_k, v_{k+1}$ be the vertices on the boundary of B, with $v_0 = v_{k+1}$ the cut vertex of B. A vertex v_i is called an *apex vertex of B with respect to $\Gamma_{\mathcal{C}/B}$ and l* if all inter-cluster neighbors of the vertices in $v_1, \ldots v_{i-1}$ are to left of l in $\Gamma_{\mathcal{C}/B}$ and the inter-cluster neighbors of the vertices $v_{i+1}, \ldots, v_k$ are to the right of l in $\Gamma_{\mathcal{C}/B}$.

▶ **Lemma 3.4.** *Let B be a free leaf block of a clustered graph $\mathcal{C}$ with an embedding ψ. Let $\Gamma_{\mathcal{C}/B}$ be a planar straight-line drawing homeomorphic to the induced embedding of $\mathcal{C}/B$ and let l be the supporting line of a U_b-similar segment. Then there is an apex vertex of B with respect to $\Gamma_{\mathcal{C}/B}$ and l.*

Proof sketch. Since $\Gamma_{\mathcal{C}/B}$ is a straight-line drawing homeomorphic to the embedding induced by the contraction of B, the neighbors of b in $\mathcal{C}/B$ appear in the same clockwise order as in a clockwise traversal of all neighbors of vertices on the boundary of B in $\mathcal{C}$. Thus, the partitioning of the neighborhood of b into the left and right of l transfers to the vertices on the boundary of B. ◻

With this framework at hand, we are now able to prove that $\mathcal{C}$ has $\mathcal{D}_{\mathcal{C}}$-framed straight-line drawing, if the B-contracted clustered graph $\mathcal{C}/B$ has a $\mathcal{D}_{\mathcal{C}}$-framed straight-line drawing $\Gamma_{\mathcal{C}/B}$. Thus, let L be the set of inter-cluster neighbors to the left of the supporting-line l of a U_b-similar segment s, and let R be the corresponding set to the right of l. We obtain two polygons σ_L and σ_R by the application of Lemma 3.3. We obtain a convex polygon σ by merging σ_L and σ_R at the common side s. An apex vertex v_i splits the vertices on the boundary of B. We place the vertices $v_0, \ldots, v_{i-1}$ on the boundary of the polygon σ_L and

$v_{i+1}, \ldots, v_k$ on the boundary of σ_R. The apex v_i is placed at the end a of the U_b-similar segment $s = \overline{ba}$ where it can be connected to vertices in L and in R. Since σ is a convex polygon, we can extend this drawing to a drawing Γ of $\mathcal{C}$ by drawing the remaining vertices of B in the interior of σ with the result of Chambers et al. [5]. We get the following result.

▶ **Lemma 3.5.** *Let $\mathcal{C} = (G, \mathcal{V})$ be a connected simple clustered graph with an embedding ψ that is compatible with a disk arrangement $\mathcal{D}_{\mathcal{C}}$. If B is a free leaf block of $\mathcal{C}$ and $\mathcal{C}/B$ has $\mathcal{D}_{\mathcal{C}}$-framed straight-line drawing homeomorphic to the embedding induced by the contraction of B, then $\mathcal{C}$ has a $\mathcal{D}_{\mathcal{C}}$-framed straight-line drawing.*

Note that, if every cluster contains exactly one vertex, then the center points of the disks in the planar disk arrangement $\mathcal{D}_{\mathcal{C}}$ induce a planar straight-line drawing of $\mathcal{C}$. Thus, we can inductively apply the previous lemma to prove our main theorem.

▶ **Theorem 3.6.** *Every connected simple clustered graph $\mathcal{C} = (G, \mathcal{V})$ with a planar embedding ψ that is compatible with a disk arrangement $\mathcal{D}_{\mathcal{C}}$ has a $\mathcal{D}_{\mathcal{C}}$-framed straight-line drawing that is homeomorphic to ψ.*

4 Conclusion

We proved that every clustered planar graph with an embedding compatible with a planar disk arrangement has a $\mathcal{D}_{\mathcal{C}}$-framed straight-line drawing. If the requirement of the disk arrangement to be planar is dropped, not every clustered-planar graph has $\mathcal{D}_{\mathcal{C}}$-framed straight-line drawing. Thus, we ask what is the complexity of deciding whether a clustered planar embedded graph has $\mathcal{D}_{\mathcal{C}}$-framed straight-line drawing for a given non-planar disk arrangement $\mathcal{D}_{\mathcal{C}}$?

References

1 Hugo A. Akitaya, Radoslav Fulek, and Csaba D. Tóth. Recognizing Weak Embeddings of Graphs. In Artur Czumaj, editor, *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'18)*, pages 274–292. Society for Industrial and Applied Mathematics, 2018. doi:10.1137/1.9781611975031.20.

2 Md. Jawaherul Alam, Michael Kaufmann, Stephen G. Kobourov, and Tamara Mchedlidze. Fitting Planar Graphs on Planar Maps. *Journal of Graph Algorithms and Applications*, 19(1):413–440, 2015. doi:10.7155/jgaa.00367.

3 Patrizio Angelini, Fabrizio Frati, and Michael Kaufmann. Straight-Line Rectangular Drawings of Clustered Graphs. *Discrete & Computational Geometry*, 45(1):88–140, 2011. doi:10.1007/s00454-010-9302-z.

4 Thomas Bläsius and Ignaz Rutter. A new perspective on clustered planarity as a combinatorial embedding problem. *Theoretical Computer Science*, 609(2):306 – 315, 2016. doi:10.1016/j.tcs.2015.10.011.

5 Erin W. Chambers, David Eppstein, Michael T. Goodrich, and Maarten Löffler. Drawing Graphs in the Plane with a Prescribed Outer Face and Polynomial Area. *Journal of Graph Algorithms and Applications*, 16(2):243–259, 2012. doi:10.7155/jgaa.00257.

6 Peter Eades, Qingwen Feng, Xuemin Lin, and Hiroshi Nagamochi. Straight-Line Drawing Algorithms for Hierarchical Graphs and Clustered Graphs. *Algorithmica*, 44(1):1–32, 2006. doi:10.1007/s00453-004-1144-8.

7 Qing-Wen Feng, Robert F. Cohen, and Peter Eades. Planarity for Clustered Graphs. In Paul Spirakis, editor, *Proceedings of the Third Annual European Symposium on Algorithms (ESA'95)*, pages 213–226. Springer Berlin/Heidelberg, 1995. doi:10.1007/3-540-60313-1_145.