A New Lower Bound on the Maximum Number of Plane Graphs using Production Matrices

Clemens Huemer[1], Alexander Pilz[2], and Rodrigo I. Silveira[1]

1 **Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Spain.**
`{clemens.huemer,rodrigo.silveira}@upc.edu`

2 **Department of Computer Science, ETH Zürich, Switzerland.**
`alexander.pilz@inf.ethz.ch`

Abstract

We use the concept of production matrices to show that there exist sets of n points in the plane that admit $\Omega(41.77^n)$ crossing-free geometric graphs. This improves the previously best known bound of $\Omega(41.18^n)$ by Aichholzer et al. (2007).

1 Introduction

A *geometric graph* on a set S of n labeled points in the Euclidean plane is a graph with vertex set S in which an edge is represented by a straight line segment between the corresponding vertices. In this work, we are interested in the number of *crossing-free* geometric graphs on a set of n points, i.e., geometric graphs in which all segments are interior-disjoint. It is easy to see that, for any n points, this number is at least exponential in n. In 1982, Ajtai et al. [2] showed that the upper bound on this number is also exponential. Currently, it is known that any set of n points admits not more than $O(187.53^n)$ crossing-free graphs [13]. While it is known that the number of crossing-free graphs is minimized if the point set is in convex position [1], not much is known about sets maximizing this number. The best known example by now is the so-called *double-zig-zag chain* [1], with $\Omega(41.18^n)$ crossing-free graphs. As usual, such lower-bound constructions rely on describing a family of point sets with convenient structural properties. In this paper, we improve this bound by showing that another well-known family of point sets, a generalization of the double-zig-zag chain, admits $\Omega(41.77^n)$ crossing-free graphs. This generalization has also been used for similar bounds on triangulations [5], but the number of general crossing-free graphs on this configuration was not known. The method that allows us to analyze these point sets is the use of *production matrices*, which we consider interesting on its own.

This method works by implicitly arranging the graphs in a *generating tree*, describing a rule to produce a graph from one on fewer points. Consider a partition of the set of graphs on $i \leq n$ points into n parts according to their degree at a special *root vertex*, and represent the cardinality of each part in a vector $\vec{v}^i$. The first element of $\vec{v}^i$ is the number of graphs with the root vertex having degree 0, the second one that of graphs with root vertex with degree 1, and so on. We then devise how to generate graphs on $i + c$ points with a new root

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vertex from the graphs counted in $\vec{v}^i$, and again give the cardinalities of their parts in a vector $\vec{v}^{i+c}$ (for some small positive number c). Our point sets will allow us to devise an $n \times n$ *production matrix* A such that $\vec{v}^{i+c} = A\vec{v}^i$. We obtain the number of graphs on n vertices in $\vec{v}^n = A^j \vec{v}^{n_0}$ from the graphs on a constant number n_0 of vertices, with $j = (n - n_0)/c$. We can then use the Perron–Frobenius theorem to obtain a lower bound on the elements of A^j when j tends to infinity by approximating the largest eigenvalue of the matrix. This gives us a lower bound on the number of crossing-free graphs on such a point set.

For points in convex position, generating trees have been described for triangulations [10], spanning trees [6], and other crossing-free graphs [7]. They are the basis of the ECO method [3]. The term *production matrix* was introduced in [4], a similar concept is known as *AGT matrix* [11]. Together with Seara, the authors already addressed characteristic polynomials of production matrices for geometric graphs [8].

In the next section, we define the family of point sets used, and provide production matrices to count subgraphs in its different parts. In Section 3, we argue that bounds on the Perron roots of the matrices give us a lower bound on the number of crossing-free graphs.

2 Generalized double zig-zag chains and the new lower bound

Basically, our point sets will be described as sequences $(s_1, \ldots, s_n)$. Consider any graph G drawn on the first $i + 1$ vertices. If we replace every edge $s_j s_{i+1}$ by the edge $s_j s_i$ for all $j \leq i + 1$ (and disregard duplicates and loops), we obtain a graph G' that we call the *parent* of G. Our sets will be such that G' is crossing-free. In the other direction, we can select some edges incident to s_i in G' and replace them by edges incident to s_{i+1} in a way that G' is the parent of the new graph $\tilde{G}$, and such that $\tilde{G}$ is crossing-free. We say that G' *produces* $\tilde{G}$, and the edges incident to s_{i+1} are *inherited*. The degree of s_i in G determines how many graphs can be produced from it. For our construction, s_i is thus the root vertex, and the vector $\vec{v}^i$ contains the number of graphs with root vertex s_i of degree j, for $0 \leq j \leq n$. While this captures the basic idea of our proofs, we will actually have to use more involved constructions, in which we add a constant number of points at once and add edges, some inherited, and some not, in a well-defined, local way.

2.1 The generalized double-zig-zag chain

Let Z_k be a set of $n = 2z$ points with $z \equiv 1 \pmod{(k+1)}$ that is arranged in the following way. Consider two x-monotone circular arcs facing each other as in Fig. 1, such that each point on one arc can *see* each point on the other arc (where two points can see each other if the interior of the line segment connecting them does not intersect one of the arcs). On each arc, we place $\lceil n/(k+1) \rceil$ points. Consider the segment between two consecutive such points s and t on the lower arc. We now place a "flat" circular arc between s and t with circle center above the arc, and place k points on it; here, flat means that moving the center of the arc up (and thus the k points on it) does not change the set of crossing-free graphs drawable on Z_k. We call the group formed by s, t, and the k points in between them a *pocket*. We place k such points between each pair of consecutive points of the lower arc (obtaining the *lower chain*), and also in an analogous way on the upper arc (resulting in the *upper chain*). See Figure 1 for an example of Z_2, where each pocket consists of four points.

The points along the lower arc, including pockets, are labeled, from left to right, $p_1, \ldots, p_z$, and those on the upper arc $q_1, \ldots, q_z$. Observe that the segment between any two consecutive points $p_i p_{i+1}$ is not crossed by any other segment between two points of the set, and thus can co-exist with any other edge in a crossing-free graph. For this reason, these edges will be

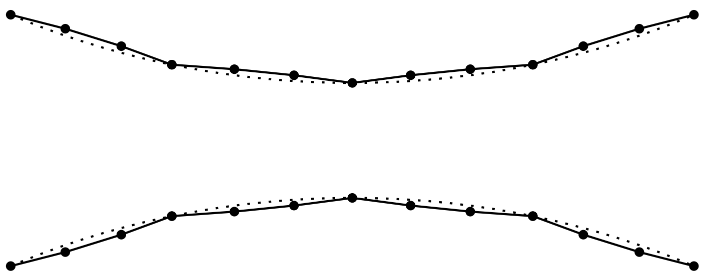

Figure 1 A generalized double-zig-zag chain Z_2. The arcs for the construction are dotted, the solid edges are not crossed by any segment between two points.

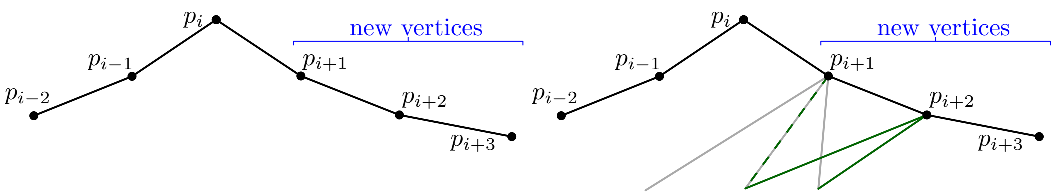

Figure 2 Part of an almost convex chain with two interior vertices (i.e., $k = 2$). Vertices p_{i-2} and p_{i+1} are leading vertices. The other vertices are regular. Since p_{i+2} is a regular vertex, any edge incident to p_{i+2} present in a plane graph can be obtained by inheriting an edge from the previous vertex p_{i+1}. The example shows p_{i+2} inheriting two edges from p_{i+2}. The last inherited edge (dashed) may also be kept at p_{i+1} without influencing the degree of p_{i+2}.

disregarded first in our counting, and will be considered in the end by multiplying by a factor of 2^n. Also note that the construction consists of two almost convex polygons [9]. Therefore we focus on counting the graphs with edges below the path $(p_1, \ldots, p_z)$ (and, symmetrically, above the path $(q_1, \ldots, q_z)$) called the *outer part*, and edges which connect vertices of the two paths, which are in the *inner part*. Our bound is obtained on Z_2.

2.2 Production matrices for the outer part

In this section we deduce matrices to count the number of plane graphs with edges below the path $(p_1, \ldots, p_z)$, as in Figure 2. Recall that a chain is composed of a series of pockets; each pocket forms a reflex chain of four vertices. The first and last vertices are convex, while the two middle ones are reflex. The first (say, with smallest index) reflex vertex is called the *leading* vertex of the chain. All other vertices we call *regular*.

We will present a matrix to count the number of plane graphs after adding one whole pocket. This matrix will be the product of three matrices, one related to each new vertex of the pocket $p_{i+1}, p_{i+2}, p_{i+3}$ (recall that p_i coincides with the last vertex of the previous pocket).

2.2.1 Matrix for regular vertices

Consider a regular vertex like p_{i+2} (refer to Figure 2). Assume that the vector $\vec{v}^{i+1}$, containing the number of plane graphs for each possible degree of p_{i+1}, is known. The plane graphs where p_{i+2} has degree 0 are equal to all the graphs counted in $\vec{v}^{i+1}$. This gives a first row of 1s in the matrix. If p_{i+2} has degree 1, it needs to inherit one edge from p_{i+1}. If the degree of p_{i+1} is 0, this is not possible, thus we get a zero in the first column of the second row. As soon as p_{i+1} has degree at least 1, p_{i+2} can inherit one edge from p_{i+1}. Moreover, there is

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 3 & 4 & 3 & 3 & 3 & 3 \\ 1 & 4 & 5 & 4 & 4 & 4 \\ 0 & 1 & 4 & 5 & 4 & 4 \\ 0 & 0 & 1 & 4 & 5 & 4 \end{pmatrix}$$

Table 1 Matrices for computing the outer part, for $n = 6$.

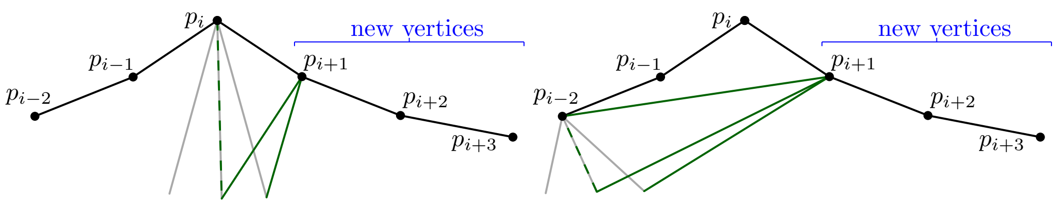

Figure 3 When edges $p_{i+1}p_{i-2}$ and $p_{i+1}p_{i-3}$ are not included, p_{i+1} can inherit edges from p_i (left). The example shows p_{i+1} inheriting two edges from p_i. The last inherited edge (dashed) may be kept without influencing the degree of p_{i+1}. The case when edges $p_{i+1}p_{i-2}$ or $p_{i+1}p_{i-3}$ are included is shown to the right. In the example $p_{i+1}p_{i-2}$ is included, and p_{i+1} inherits two edges from p_{i-2}. The dashed edge can be optionally kept.

the option of keeping (a copy of) the inherited edge incident to p_{i+1} without creating any crossing. In total, for each graph in which p_{i+1} has degree at least one, that gives two ways for making p_{i+2} have degree 1. Thus the rest of the row is made of 2s.

The following rows are analogous, shifted by one column every time: in order for p_{i+2} to have degree k, k edges need to be inherited from p_{i+1}, thus the minimum degree for p_{i+1} is k. Since we can always choose to keep the last inherited edge incident to p_{i+1}, we get 2 options every time. This results in matrix R in Table 1. Exactly the same matrix applies to p_{i+3}.

2.2.2 Matrix for leading vertices

Leading vertices like p_{i+1} in Figure 2 require a different approach, as there are edges incident to p_{i+1} that cannot be obtained by inheriting from p_i (i.e., edges $p_{i+1}p_{i-1}$, $p_{i+1}p_{i-2}$, $p_{i+1}p_{i-3}$, as p_ip_{i-1}, p_ip_{i-2}, p_ip_{i-3} are not in the outer part). To take this into account, we consider two cases, depending on whether edges $p_{i+1}p_{i-2}$ or $p_{i+1}p_{i-3}$ are included or not.

Case 1: Edges $\boldsymbol{p_{i+1}p_{i-2}}$ and $\boldsymbol{p_{i+1}p_{i-3}}$ are not included. When $p_{i+1}p_{i-2}$ and $p_{i+1}p_{i-3}$ are not included, p_{i+1} can inherit edges from p_i (notice that all edges from p_i cross $p_{i+1}p_{i-2}$ and $p_{i+1}p_{i-3}$). See Figure 3 (left).

The plane graphs where p_{i+1} has degree zero are, as before, all the ones counted in $\vec{v}^i$, thus this gives a first row of 1s in the matrix. If p_{i+1} has degree one, it either inherited one edge from p_i or is connected to p_{i-1}. Thus if p_i has degree zero, there is only one possibility: using edge $p_{i+1}p_{i-1}$. That gives a 1 in the first column of the second row. As soon as p_i has degree at least one, p_{i+1} can inherit one edge from p_i, with the additional option of keeping the inherited edge incident to p_i; note that in this case using also $p_{i+1}p_{i-1}$ is not considered because that would increase the degree of p_{i+1} by one. In total, for each possible degree of p_i, that gives two ways for making p_{i+1} have degree one. Thus the rest of the row is made of 2s.

The following rows are analogous. Consider the kth row ($k \geq 3$). If p_i has degree $k-2$ or less, it is impossible for p_{i+1} to obtain degree k. When p_i has degree $k-1$, there is one possibility: to inherit all edges incident to p_i and add edge $p_{i+1}p_{i-1}$. If p_i has degree at

$$Q = \begin{pmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 10 & 6 & 0 & 0 & 0 & 0 \\ 5 & 10 & 6 & 0 & 0 & 0 \\ 1 & 5 & 10 & 6 & 0 & 0 \\ 0 & 1 & 5 & 10 & 6 & 0 \\ 0 & 0 & 1 & 5 & 10 & 6 \end{pmatrix} \quad F = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Table 2 Matrices for computing the inner part, for $n = 6$.

least k, then p_{i+1} can inherit k edges from p_i, with the additional option of keeping the last inherited edge incident to p_i, giving two options for every possible degree of p_i. This leads to the matrix C in Table 1.

Case 2: At least one of $p_{i+1}p_{i-2}$ and $p_{i+1}p_{i-3}$ is included. In this case we proceed essentially by using p_{i-2} as "previous" vertex, but considering also the three special edges $\{p_{i+1}p_{i-1}, p_{i+1}p_{i-2}, p_{i+1}p_{i-3}\}$, which cannot be inherited from p_{i-2}. Refer to Figure 3 (right). We defer the details to the full version. The result for this case is matrix X, shown in Table 1.

2.3 Production matrices for the inner part

The number of graphs on the inner part can be bounded similar to [1]. However, in the full version, we show how to obtain a lower bound using production matrices, based on two additional matrices, Q and F shown in Table 2.

2.4 Putting things together

The final production matrix for the outer part is obtained by combining matrices R, C, and X. For each of the two regular vertices it is enough to multiply the previous vector by R. For the leading vertex we need to combine the two cases, thus we need to add up C and X. However, the reasoning in X uses p_{i-2} instead of the previous vertex p_i. Thus prior to multiplying by X, we need to recover the vector corresponding to p_{i-2}: for this we first multiply twice by R^{-1}. Thus the final combined matrix for the outer part is $R^2(C + X \cdot R^{-2})$. In the full version we show that a lower bound on the number of plane graphs in the inner part is given by the combined matrix $(FFR + 2R)Q$.

3 A lower bound using the eigenvalue

All our production matrices are non-negative. The zero entries are exactly those below a sub-diagonal. Thus, they are irreducible and primitive (Frobenius' test for primitivity holds, cf. [12, p. 678]). Let A be a production matrix of fixed size $m \times m$. We know therefore that

$$\lim_{n\to\infty} \left(\frac{A}{r}\right)^n = \frac{\vec{p}\vec{q}^T}{\vec{q}^T\vec{p}} > 0 \ ,$$

where $\vec{p}$ and $\vec{q}$ are the Perron vectors of A and A^T, respectively, and r is the Perron root (i.e., largest eigenvalue) of A [12, p. 674]. As these values are constant and each entry of A^n is in $\Theta(r^n)$, this provides a means of obtaining the asymptotic number of elements constructed by the production matrix: multiplying the initial degree vector with A^i gives the degree vector for $ci < m$ points. However, there is one caveat. The exponent n tends to infinity, and we thus cannot use this to argue about matrices of size n. The matrix size must be fixed. However, for obtaining lower bounds, we can take the nth power of a $(m \times m)$ production matrix for some constant m to obtain a lower bound on the number of graphs on n vertices. In the first iteration where we add a point larger than the size of the matrix, we do not count some graphs with high degree at the last point. These are also not taken into account in the next iteration etc., where we also produce graphs of smaller degree at the last point. Still, the degree vector gives a lower bound on the number of graphs. We may thus obtain the Perron root r of a constant-size production matrix and know that the number of graphs on n vertices in that class is in $\Omega(r^n)$ for all our considered instances. For the matrix $R^2(C+X\cdot R^{-2})$, the largest eigenvalue is at least 124.22239555, when taking the constant-size production matrix large enough. For the inner part, the largest eigenvalue of the matrix $(FFR+2R)Q$ is at least 5380.90657056 (see the full version). Accounting for the 2^n ways to add edges along the chains, we get $\Omega((\sqrt[3]{124.22239555}\cdot\sqrt[6]{5380.90657056}\cdot 2)^n) = \Omega(41.773981586^n)$ crossing-free graphs (eigenvalues computed using *Mathematica 11.2* with $m = 1024$).

4 Conclusion

We slightly improved the current lower bound on the maximum number of crossing-free geometric graphs on n points using production matrices. Applying production matrices to families of well-structured point sets appears to be an easy way of obtaining bounds for certain types of graphs (e.g., triangulations). It is also easy to mix the pocket sizes. However, our current approach results in an increasing number of cases when considering generalized double-zig-zag chains with larger pockets.

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