Coxeter triangulations have good quality

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Abstract

Coxeter triangulations are triangulations of Euclidean space based on a single simplex. By this we mean that given an individual simplex we can recover the entire triangulation of Euclidean space by inductively reflecting in the faces of the simplex. In this paper we establish that the quality of the simplices in all Coxeter triangulations is $O(1/\sqrt{d})$ of the quality of regular simplex. We further investigate the Delaunay property (and an extension thereof) for these triangulations. In particular, one family of Coxeter triangulations achieves the protection $O(1/d^2)$. We conjecture that both bounds are optimal for triangulations in Euclidean space.

1 Introduction

1.1 Motivation and related work

Well shaped simplices are of importance for various fields of application such as finite element methods and manifold meshing. Poorly-shaped simplices may induce various problems in finite element method, such as large discretization errors or ill-conditioned stiffness matrices. A simplex is well shaped if its quality is good, which can be expressed in terms of various *quality measures*. Some examples of quality measures are: the ratio between minimal height and maximal edge length ratio called *thickness*, the ratio between volume and a power of the maximal edge length called *fatness*, and the *inradius-circumradius ratio*. Bounds on *dihedral angles* can also be included in the list of quality measures. We stress that there are many other quality measures in use and authors often find useful to introduce measures that are specific to whatever problem they study. Finding triangulations, even in Euclidean space, of which all simplices have good quality is a non-trivial exercise in arbitrary dimension.

In this paper we shall discuss Coxeter triangulations.

▶ **Definition 1.1.** A monohedral[1] triangulation is called *Coxeter triangulation* if all its d-simplices can be obtained by consecutive reflections through facets of the d-simplices in the triangulation.

There are four families of Coxeter triangulations and five exceptional ones. All three two-dimensional Coxeter triangulations and the simplices of the three three-dimensional Coxeter triangulations are illustrated in Figure 1. For an extended introduction we refer to the pioneering paper on reflection groups by Coxeter [10] and the classical book on Lie groups and algebras by Bourbaki [5]. Another classical reference book is “Sphere packings, Lattices, and Groups” by Conway and Sloane [9].

To our knowledge, these are the triangulations with the best quality in arbitrary dimension. In particular, all dihedral angles of simplices in Coxeter triangulations are 45°, 60° or 90°, with the exception of the so-called $\tilde{G}_2$ triangulation of the plane where we also can find an

[1] A triangulation of $\mathbb{R}^d$ is called *monohedral* if all its d-simplices are congruent.

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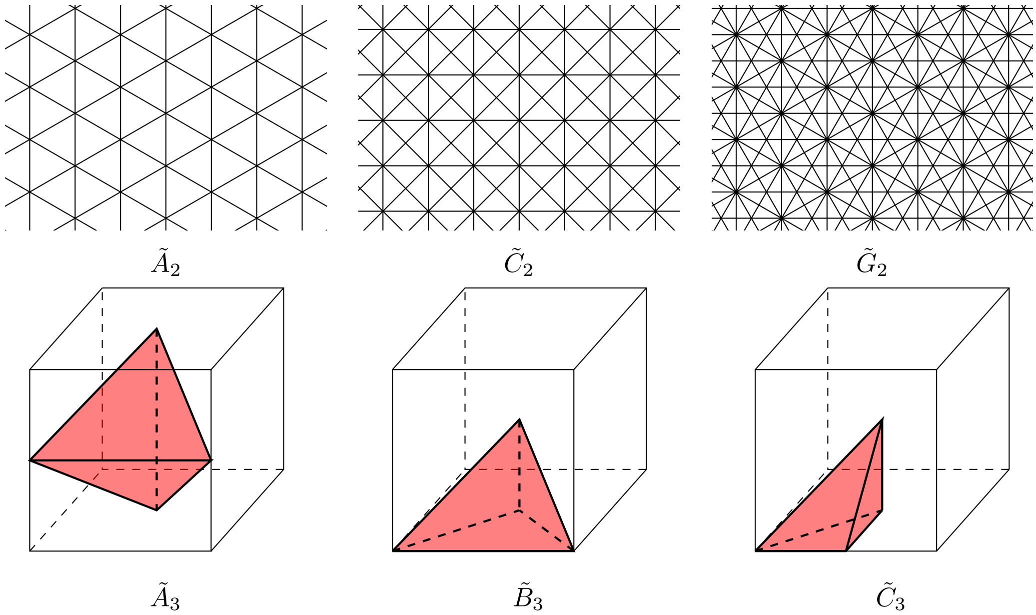

Figure 1 Above: Coxeter triangulations in $\mathbb{R}^2$. Below: simplices of Coxeter triangulations in $\mathbb{R}^3$ represented as a portion of a cube.

angle of 30°. This is a clear sign of the exceptional quality of the simplices involved. Our goal is to exhibit the extraordinary properties of Coxeter triangulations and promote their use in the Computational Geometry community.

The notion of Coxeter triangulations was introduced to the computational geometry community by Dobkin, Wilks, Levy and Thurston in [11], where they tackled the problem of contour-tracing in $\mathbb{R}^d$. The choice of Coxeter triangulations was motivated by the following requirements:

- It should be easy to find the simplex that shares a facet with a given simplex.
- It should be possible to label the vertices of all the simplices at the same time with indices $0, \ldots, d$, in such a way that each of the $d+1$ vertices of a d-simplex has a different label.
- The triangulations should be monohedral, meaning that all simplices are congruent.
- All simplices should be isotropic, meaning that they should be roughly the same in all directions.

Coxeter triangulations exactly fit these requirements. After comparing the inradius-circumradius ratio of the simplices in the triangulations Dobkin et al. chose the $\tilde{A}_d$ Coxeter triangulation as the underlying triangulation for their contour-tracing algorithm.

The same $\tilde{A}_d$ Coxeter triangulation appeared in the works by Adams, Baek and Davis [1] and Choudhary, Kerber and Raghvendra [7], among others.

The three-dimensional Coxeter triangulation $\tilde{A}_3$ has attracted attention in the 3D mesh generation community for the high-quality of its simplices. The vertex set of this triangulation is also known as the *body-centred cubic lattice*, or bcc lattice, and its tetrahedron is sometimes referred as *Sommerville's type II tetrahedron* or simply bcc-tetrahedron. This tetrahedron has been shown to be the best-conditioned space-filling tetrahedron out of all space-filling tetrahedra used in the 3D mesh community by a number of conditioning measures.

Apart from quality, we are also interested in the stronger requirement of *protection* [2], which is specific to Delaunay triangulations. It has been proven that protection guarantees good quality [2]. Some algorithms were introduced for the construction of a protected set, such as the perturbation-based algorithms in [3] and [4]. Both of these algorithms take a general ε-net in $\mathbb{R}^d$ as input and output a δ-protected net with δ of the order just $\Omega(2^{-d^2}\varepsilon)$. The d-dimensional Coxeter triangulation $\tilde{A}_d$ provides us another extremity. As we will see in the following, this highly-structured triangulation is Delaunay with protection $O(\frac{1}{d^2}\varepsilon)$. This protection value is the greatest in a general d-dimensional Delaunay triangulation we know.

The Coxeter triangulations we study are intricately linked with root systems and root lattices. Delaunay triangulations of the root lattices have been studied by Conway and Sloane [9] and Moody and Patera [12]. These triangulations are different from the ones we study: the vertex sets we use are not necessarily lattices (see Theorem 4.1).

1.2 Contribution

In this paper we give explicit expressions of a number of quality measures of Coxeter triangulations for all dimensions, presented in Section 4. This is an extension of the work by Dobkin et al. [11] who presented a table of the values of the inradius-circumradius ratio for the Coxeter triangulations up to dimension 8. We also provide explicit measures of the corresponding simplices in the full version of the current paper [8, Appendix B], allowing the reader to compute quality measures other than the ones listed.

In Section 2, we state the theorem of optimality of the regular d-simplex for each of the chosen quality measures. This theorem justifies the definition of the normalized versions of these quality measures.

In Section 3, we established a criterion to identify if any given monohedral triangulation is Delaunay.

The proofs of the statements can be found in the full version [8], as well as extra introductory material.

1.3 Future work

The simplex qualities, defined in Definition 2.1, of the four families of Coxeter triangulations behave as $O(1/\sqrt{d})$ in terms of dimension. We conjecture that this quality is optimal for a general space-filling triangulation in $\mathbb{R}^d$. In addition, the d-dimensional Coxeter triangulation $\tilde{A}_d$ has the relative Delaunay protection $O(1/d^2)$. We further conjecture that it is optimal for a general space-filling triangulation in $\mathbb{R}^d$. These conjectures are motivated by the extraordinary lower and upper bounds on the dihedral angles of simplices in Coxeter triangulations; they are precisely 45°, 60° or 90° for the four families. Moreover the circumcentres of the simplices of $\tilde{A}_d$ lie very far inside the simplices.

2 Quality definitions

The quality measures we are interested in, we call *aspect ratio*, *fatness*, *thickness* and *radius ratio*. Their formal definitions are as follows:

▶ **Definition 2.1.** Let $h(\sigma)$ denote the minimal height, $r(\sigma)$ the inradius, $R(\sigma)$ the circumradius, $vol(\sigma)$ the volume and $L(\sigma)$ the maximal edge length of a given d-simplex σ.

- The *aspect ratio* of σ is the ratio of its minimal height to the diameter of its circumscribed ball: $\alpha(\sigma) = \frac{h(\sigma)}{2R(\sigma)}$.
- The *fatness* of σ is the ratio of its volume to its maximal edge length taken to the power d: $\Theta(\sigma) = \frac{vol(\sigma)}{L(\sigma)^d}$.

- The *thickness* of σ is the ratio of its minimal height to its maximal edge length: $\theta(\sigma) = \frac{h(\sigma)}{L(\sigma)}$.
- The *radius ratio* of σ is the ratio of its inradius to its circumradius: $\rho(\sigma) = \frac{r(\sigma)}{R(\sigma)}$.

To be able to compare the presented quality measures between themselves, we will normalize them by their respective maximum value. As we show, all of these quality measures are maximized by regular simplices.

▶ **Theorem 2.2.** *Out of all d-dimensional simplices, the regular d-simplex has the highest aspect ratio, fatness, thickness and radius ratio.*

For a quality measure κ we will define the normalized quality measure $\hat{\kappa}$, such that for each d-simplex σ, $\hat{\kappa}(\sigma) = \frac{\kappa(\sigma)}{\kappa(\Delta)}$, where Δ is the regular d-simplex. Theorem 2.2 ensure that the quality measures $\hat{\rho}$, $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\Theta}$ take their values in $[0, 1]$ surjectively.

3 Delaunay criterion for Coxeter triangulations

Many of the provably good mesh generation algorithms are based on Delaunay triangulations [6]. This motivated us to investigate if Coxeter triangulations have the Delaunay property. We established the following criterion, extending the work by Rajan [13] on triangulations consisting of self-centered simplices.

▶ **Definition 3.1.** A simplex is called *self-centred* if it contains its circumcentre inside or on the boundary.

▶ **Theorem 3.2.** *A Coxeter triangulation is Delaunay if and only if its simplices are self-centred.*

Because some of the triangulations that interest us here are Delaunay, we will also look at their *protection* value.

▶ **Definition 3.3.** The *protection* of a d-simplex σ in a Delaunay triangulation on a point set P is the minimal distance of points in $P \setminus \sigma$ to the circumscribed ball of σ:

$$\delta(\sigma) = \inf_{p \in P \setminus \sigma} d(p, B(\sigma)), \text{ where } B(\sigma) \text{ is the circumscribed ball of } \sigma.$$

The *protection* δ of a Delaunay triangulation $\mathcal{T}$ is the infimum over the d-simplices of the triangulation: $\delta = \inf_{\sigma \in \mathcal{T}} \delta(\sigma)$. A triangulation with a positive protection is called *protected*.

We define the *relative protection* $\hat{\delta}(\sigma)$ of a given d-simplex σ to be the ratio of the protection to its circumscribed radius: $\hat{\delta}(\sigma) = \frac{\delta(\sigma)}{R(\sigma)}$.

The *relative protection* $\hat{\delta}$ of a Delaunay triangulation $\mathcal{T}$ is the infimum over the d-simplices of the triangulation: $\hat{\delta} = \inf_{\sigma \in \mathcal{T}} \hat{\delta}(\sigma)$. We can determine if a Coxeter triangulation is not protected with the help of the following theorem.

▶ **Theorem 3.4.** *1. A Delaunay triangulation of $\mathbb{R}^d$ where a simplex with maximal circumradius contains the circumcentre on its boundary is not protected.*

2. If a simplex of a Coxeter triangulation contains the circumcentre on its boundary, then the triangulation is non-protected Delaunay.

3. If a simplex of a Coxeter triangulation contains the circumcentre strictly inside, then the triangulation is Delaunay with a non-zero protection.

4 Main result

In this section we present a table with explicit expressions of quality measures of Coxeter triangulations. In addition to that we also identify which Coxeter triangulations are Delaunay and give their protection values. Finally, we identify which Coxeter triangulations have vertex sets with lattice structure.

▶ **Theorem 4.1.** *The normalized fatness, aspect ratio, thickness and radius ratio of simplices in Coxeter triangulations, as well as Delaunay property are:*

	Fatness $\hat{\Theta}^{1/d}$	*Aspect Ratio* $\hat{\alpha}$	*Thickness* $\hat{\theta}$	*Radius Ratio* $\hat{\rho}$	*Delaunay?*
$\tilde{A}_d$, d odd	$\frac{2^{3/2}}{\left(\sqrt{d+1}\right)^{1+2/d}}$	$\sqrt{\frac{6d}{(d+1)(d+2)}}$	$\frac{2\sqrt{d}}{d+1}$	$\sqrt{\frac{6d}{(d+1)(d+2)}}$	✓
$\tilde{A}_d$, d even	$\frac{2^{3/2}\left(\sqrt{d+1}\right)^{1-2/d}}{\sqrt{d(d+2)}}$		$\frac{2}{\sqrt{d+2}}$		
$\tilde{B}_d$	$\frac{2^{1/2+1/d}}{\sqrt{d}\left(\sqrt{d+1}\right)^{1/d}}$	$\frac{d\sqrt{2}}{(d+1)\sqrt{d+2}}$	$\frac{1}{\sqrt{d+1}}$	$\frac{2d}{\sqrt{d+2}(1+(d-1)\sqrt{2})}$	✗
$\tilde{C}_d$	$\frac{\sqrt{2}}{\sqrt{d}\left(\sqrt{d+1}\right)^{1/d}}$	$\frac{\sqrt{2d}}{d+1}$	$\frac{1}{\sqrt{d+1}}$	$\frac{2\sqrt{d}}{2+(d-1)\sqrt{2}}$	✓
$\tilde{D}_d$	$\frac{2^{1/2+2/d}}{\sqrt{d}\left(\sqrt{d+1}\right)^{1/d}}$	$\frac{d\sqrt{2}}{(d+1)\sqrt{d+4}}$	$\frac{1}{\sqrt{d+1}}$	$\frac{d\sqrt{2}}{(d-1)\sqrt{d+4}}$	✗
$\tilde{E}_6$	$\sqrt[12]{\frac{64}{137781}}$	$\frac{2}{7}$	$\frac{1}{\sqrt{14}}$	$\frac{1}{2}$	✗
$\tilde{E}_7$	$\sqrt[14]{\frac{1}{177147}}$	$\frac{7\sqrt{13}}{104}$	$\frac{\sqrt{21}}{24}$	$\frac{14\sqrt{13}}{117}$	✗
$\tilde{E}_8$	$\sqrt[8]{\frac{1}{3240}}$	$\frac{8\sqrt{19}}{171}$	$\frac{2\sqrt{19}}{57}$	$\frac{8\sqrt{19}}{95}$	✗
$\tilde{F}_4$	$\sqrt[8]{\frac{1}{405}}$	$\frac{4\sqrt{2}}{15}$	$\frac{2\sqrt{5}}{15}$	$\frac{4\sqrt{2}}{3(2+\sqrt{2})}$	✓
$\tilde{G}_2$	$\frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{2}{1+\sqrt{3}}$	✓

Out of them, only $\tilde{A}$ family triangulations have a non-zero relative protection value equal to:

$$\hat{\delta} = \frac{\sqrt{d^2+2d+24} - \sqrt{d^2+2d}}{\sqrt{d^2+2d}} \sim \frac{12}{d^2}$$

Only $\tilde{A}$ family, $\tilde{C}$ family and $\tilde{D}_4$ triangulations have vertex sets with lattice structure.

For the proof, refer to the full version [8]. The corresponding quality measures for the regular d-simplex Δ (which does not correspond to a triangulation in general) are:

	Fatness Θ	Aspect Ratio α	Thickness θ	Radius Ratio ρ
Δ	$\frac{1}{d!}\sqrt{\frac{d+1}{2^d}}$	$\frac{d+1}{2d}$	$\sqrt{\frac{d+1}{2d}}$	$\frac{1}{d}$

All simplex quality measures in the table above are normalized with respect to the regular simplex. Note that the fatness values in the table are given with power $1/d$. It is due to the fact that fatness is a volume-based simplex quality, and taking the $(1/d)$-th power allows a better comparison. Also note that all normalized simplex qualities for the families $\tilde{A}_d$, $\tilde{B}_d$, $\tilde{C}_d$ and $\tilde{D}_d$ behave as $O(1/\sqrt{d})$. As illustrated for fatness and radius ratio in Figure 2, $\tilde{A}_d$ achieves the greatest simplex quality among the four families of Coxeter triangulations in each dimension d.

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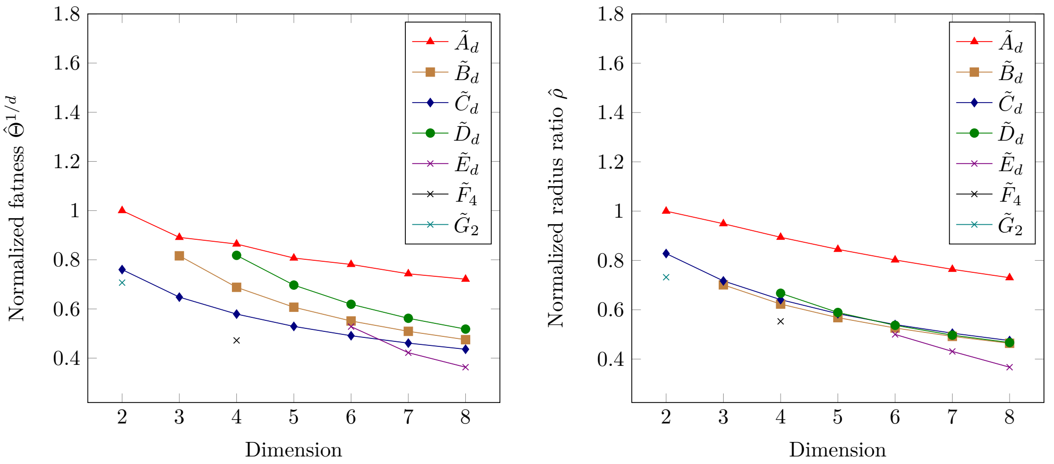

Figure 2 The visual representation of the normalized fatness and the radius ratio for simplices of $\tilde{A}_d$, $\tilde{B}_d$, $\tilde{C}_d$ and $\tilde{D}_d$ triangulations.

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