Decomposing Modalities

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Logical Frameworks and Meta-Languages:
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Logical Frameworks

- Understanding laws governing systems of logical inference
  - Semantically (models)
  - Syntactically (proofs)
  - Pragmatically (applications)
- Key concepts and techniques
  - Separating judgments from propositions
  - Hypothetical and general judgments
  - Linear hypothetical judgments
  - Categorical judgments
  - Structural cut elimination
  - Focusing and polarization
- Frontier: Modalities
  - This talk: Analyzing the fine structure of necessity
  - Vivek Nigam (11am): Subexponentials!

Defining Modalities

- Expressing different modes of truth
  - Necessary, possible
  - At time t, as known by K, ...
- Understanding modalities
  - Axiomatically [Lewis'10]
  - Semantically [Kripke'59]
  - Proof-theoretically [Prawitz'65]
    - Intuitionistically [Simpson'94] [Pf&Wong'95] [Bierman&dePaiva'96] [Davies&Pf'01]
- Not every set of axioms or accessibility relations define well-behaved logics

Applications in Computer Science

- A personal and biased sampling
- Propositions as types, proofs as programs
  - Staged computation, run-time code generation (JS4) [Davies & Pf'96]
  - Monadic encapsulation (lax logic) [Fairtlough & Mendler'97]
  - Partial evaluation (temporal logic) [Davies'96]
  - Proof irrelevance (JK) [Pf'08]
  - Message-passing concurrency (linear logic) [Caires & Pf'10] [Toninho'15]
- Reasoning about programs
  - Dynamic logic [Pratt'74]
  - Temporal logics [Pnueli'77] [Clarke & Emerson'80]
  - Separation logic [O'Hearn & Pym'99] [Reynolds'02]
- Security
  - Authorization logics [Garg et al.'06]
  - Protocol logics [Datta et al.'03]

Judging Modalities

- Axiomatics: too flexible to be decisive
- Semantics: too flexible to be decisive
- Pragmatics: applications in computer science
- Proof theory [Gentzen'35]
  - Harmony [Dummett'76]
  - Structural cut elimination [Pf'95]
- Logical frameworks [de Bruijn'68]
  - Verifications as meaning explanations [Martin-Löf'80]
  - Separating judgments from propositions [Martin-Löf'83]
  - Hypothetical/general judgments [Harper et al.'87]
  - Categorical judgments [Pf & Davies'01]
- Linear logic [Girard'87]
  - Essence of logical connectives
  - Decomposition $A \rightarrow B \simeq\, !A \multimap B$
  - Focusing [Andreoli'92]
  - Judgmental explanation [Chang et al.'03]

Aspects of Necessity

- This talk: concentrate on necessity ($\Box A$, $!A$)
  - Internalizes a categorical judgment
  - Controls weakening and contraction in linear logic
  - Corresponds to reflexivity and transitivity of accessibility relation
- How interdependent are these aspects of necessity?
- Do sensible subsystems have applications?
- Is necessity indivisible?

Outline

- An axiomatic approach to linear logic
- Judgmental sequent calculi for subsystems
- Adjoint decomposition of necessity

Implicational Linear Logic

- Intuitionistic version

$$\begin{array}{ll} \vdash (A \multimap B) \multimap (B \multimap C) \multimap (A \multimap C) & (L) \\ \vdash A \multimap A & (I) \\ \vdash (A \multimap B \multimap C) \multimap (B \multimap A \multimap C) & (X) \end{array}$$

$$\frac{\vdash A \multimap B \quad \vdash A}{\vdash B}\ \mathrm{MP}$$

Judging Axiomatic Systems

- The internal criterion for axiomatic formulations of logics is the deduction theorem for hypothetical proofs
- The external criterion will be correspondence to a sequent calculus with structural cut elimination and identity
- Start with internal criterion
  - Introduce linear hypothetical judgment
  - Prove deduction theorem
  - Illustrate how proof suggests axioms
  - Motif repeats for modal extensions

A Linear Hypothetical Hilbert System

- $\Delta ::= \bullet \mid \Delta, A$ (modulo exchange)
- $\Delta \vdash A$ is linear hypothetical judgment

$$\frac{}{A \vdash A}\ \text{HYP} \qquad \frac{\Delta_1 \vdash A \multimap B \quad \Delta_2 \vdash A}{\Delta_1, \Delta_2 \vdash B}\ \text{MP}$$

$$\frac{}{\bullet \vdash (A \multimap B) \multimap (B \multimap C) \multimap (A \multimap C)}\ L$$

$$\frac{}{\bullet \vdash A \multimap A}\ I$$

$$\frac{}{\bullet \vdash (A \multimap B \multimap C) \multimap (B \multimap A \multimap C)}\ X$$

Deduction Theorem

Theorem (Deduction)

$$\frac{\Delta, A \vdash B}{\Delta \vdash A \multimap B}\ \mathrm{DED}$$

Proof.

By induction on the deduction of $\Delta, A \vdash B$.

- Dashed line indicates an admissible rule
- Corollary

$$A_1, \ldots, A_n \vdash A \quad \text{iff} \quad \vdash A_1 \multimap \cdots \multimap A_n \multimap A$$

Proof of Deduction Theorem

Part 1.

By induction on the deduction of $\Delta, A \vdash B$.

Cases: Axioms (L), (I), or (X). Impossible, since there are no hypotheses. For example:

$$\frac{}{\bullet \vdash (A \multimap B) \multimap (B \multimap C) \multimap (A \multimap C)}\ L$$

Proof of Deduction Theorem

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Case:

$$\dfrac{}{A \vdash A}\ \text{HYP}$$

Proof of Deduction Theorem

Part 1.

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Case:

$$\dfrac{}{A \vdash A}\ \text{HYP}$$

$\bullet \vdash A \multimap A$

Proof of Deduction Theorem

Part 1.

By induction on the deduction of $\Delta, A \vdash B$.

Cases: Axioms (L), (I), or (X). Impossible, since there are no hypotheses. For example:

$$\frac{}{\bullet \vdash (A \multimap B) \multimap (B \multimap C) \multimap (A \multimap C)}\ L$$

Case:

$$\frac{}{A \vdash A}\ \text{HYP}$$

$$\bullet \vdash A \multimap A \qquad (I)$$

Proof of Deduction Theorem

Part 2.

Case:

$$\frac{\Delta_1, A \vdash B \multimap C \quad \Delta_2 \vdash B}{\Delta_1, \Delta_2, A \vdash C}\ \text{MP}$$

Proof of Deduction Theorem

Part 2.

Case:

$$\frac{\Delta_1, A \vdash B \multimap C \quad \Delta_2 \vdash B}{\Delta_1, \Delta_2, A \vdash C} \text{ MP}$$

$\Delta_1, \Delta_2 \vdash A \multimap C$

Proof of Deduction Theorem

Part 2.

Case:

$$\frac{\Delta_1, A \vdash B \multimap C \quad \Delta_2 \vdash B}{\Delta_1, \Delta_2, A \vdash C} \text{ MP}$$

$\Delta_1 \vdash A \multimap B \multimap C$ i.h.

$\Delta_1, \Delta_2 \vdash A \multimap C$

Proof of Deduction Theorem

Part 2.

Case:

$$\frac{\Delta_1, A \vdash B \multimap C \quad \Delta_2 \vdash B}{\Delta_1, \Delta_2, A \vdash C} \text{ MP}$$

$\Delta_1 \vdash A \multimap B \multimap C$ i.h.

$\Delta_2 \vdash B$ second premise

$\Delta_1, \Delta_2 \vdash A \multimap C$

Proof of Deduction Theorem

Part 2.

Case:

$$\frac{\Delta_1, A \vdash B \multimap C \quad \Delta_2 \vdash B}{\Delta_1, \Delta_2, A \vdash C}\ \text{MP}$$

$\Delta_1 \vdash A \multimap B \multimap C$ i.h.
$\bullet \vdash (A \multimap B \multimap C) \multimap (B \multimap A \multimap C)$ (X)

$\Delta_2 \vdash B$ second premise
$\Delta_1, \Delta_2 \vdash A \multimap C$

Proof of Deduction Theorem

Part 2.

Case:

$$\frac{\Delta_1, A \vdash B \multimap C \quad \Delta_2 \vdash B}{\Delta_1, \Delta_2, A \vdash C}\ \text{MP}$$

$\Delta_1 \vdash A \multimap B \multimap C$	i.h.
$\bullet \vdash (A \multimap B \multimap C) \multimap (B \multimap A \multimap C)$	(X)
$\Delta_1 \vdash B \multimap A \multimap C$	MP
$\Delta_2 \vdash B$	second premise
$\Delta_1, \Delta_2 \vdash A \multimap C$	

Proof of Deduction Theorem

Part 2.

Case:

$$\frac{\Delta_1, A \vdash B \multimap C \quad \Delta_2 \vdash B}{\Delta_1, \Delta_2, A \vdash C}\ \text{MP}$$

$\Delta_1 \vdash A \multimap B \multimap C$	i.h.
$\bullet \vdash (A \multimap B \multimap C) \multimap (B \multimap A \multimap C)$	(X)
$\Delta_1 \vdash B \multimap A \multimap C$	MP
$\Delta_2 \vdash B$	second premise
$\Delta_1, \Delta_2 \vdash A \multimap C$	MP

Proof of Deduction Theorem

Part 3.

Case:

$$\frac{\Delta_1 \vdash B \multimap C \quad \Delta_2, A \vdash B}{\Delta_1, \Delta_2, A \vdash C}\ \mathsf{MP}$$

$\square$

Proof of Deduction Theorem

Part 3.

Case:

$$\frac{\Delta_1 \vdash B \multimap C \quad \Delta_2, A \vdash B}{\Delta_1, \Delta_2, A \vdash C}\ \mathrm{MP}$$

$$\Delta_1, \Delta_2 \vdash A \multimap C$$

$\square$

Proof of Deduction Theorem

Part 3.

Case:

$$\frac{\Delta_1 \vdash B \multimap C \quad \Delta_2, A \vdash B}{\Delta_1, \Delta_2, A \vdash C}\ \mathrm{MP}$$

$\Delta_2 \vdash A \multimap B$ i.h.

$\Delta_1, \Delta_2 \vdash A \multimap C$

□

Proof of Deduction Theorem

Part 3.

Case:

$$\frac{\Delta_1 \vdash B \multimap C \quad \Delta_2, A \vdash B}{\Delta_1, \Delta_2, A \vdash C} \text{ MP}$$

$\Delta_2 \vdash A \multimap B$ i.h.

$\Delta_1 \vdash B \multimap C$ first premise

$\Delta_1, \Delta_2 \vdash A \multimap C$

□

Proof of Deduction Theorem

Part 3.

Case:

$$\frac{\Delta_1 \vdash B \multimap C \quad \Delta_2, A \vdash B}{\Delta_1, \Delta_2, A \vdash C}\ \text{MP}$$

$\Delta_2 \vdash A \multimap B$ — i.h.

$\bullet \vdash (A \multimap B) \multimap (B \multimap C) \multimap (A \multimap C)$ — (L)

$\Delta_1 \vdash B \multimap C$ — first premise

$\Delta_1, \Delta_2 \vdash A \multimap C$

□

Proof of Deduction Theorem

Part 3.

Case:

$$\frac{\Delta_1 \vdash B \multimap C \quad \Delta_2, A \vdash B}{\Delta_1, \Delta_2, A \vdash C}\ \text{MP}$$

$$\begin{array}{lr}
\Delta_2 \vdash A \multimap B & \text{i.h.} \\
\bullet \vdash (A \multimap B) \multimap (B \multimap C) \multimap (A \multimap C) & (L) \\
\Delta_2 \vdash (B \multimap C) \multimap (A \multimap C) & \text{MP} \\
\Delta_1 \vdash B \multimap C & \text{first premise} \\
\Delta_1, \Delta_2 \vdash A \multimap C &
\end{array}$$

□

Proof of Deduction Theorem

Part 3.

Case:

$$\frac{\Delta_1 \vdash B \multimap C \quad \Delta_2, A \vdash B}{\Delta_1, \Delta_2, A \vdash C}\ \text{MP}$$

$\Delta_2 \vdash A \multimap B$	i.h.
$\bullet \vdash (A \multimap B) \multimap (B \multimap C) \multimap (A \multimap C)$	(L)
$\Delta_2 \vdash (B \multimap C) \multimap (A \multimap C)$	MP
$\Delta_1 \vdash B \multimap C$	first premise
$\Delta_1, \Delta_2 \vdash A \multimap C$	MP

□

Linear Sequent Calculus

- Construct a linear sequent calculus
  - Prove structural cut elimination and identity
  - Show correspondence with Hilbert system
- $\Delta \Vdash A$ is linear hypothetical judgment
- Judgmental rules of identity and cut

$$\frac{}{A \Vdash A}\ \mathrm{id}_A \qquad \frac{\Delta_1 \Vdash A \quad \Delta_2, A \Vdash C}{\Delta_1, \Delta_2 \Vdash C}\ \mathrm{cut}_A$$

- Propositional right and left rules for $\multimap$

$$\frac{\Delta, A \Vdash B}{\Delta \Vdash A \multimap B}\ {\multimap}R \qquad \frac{\Delta_1 \Vdash A \quad \Delta_2, B \Vdash C}{\Delta_1, \Delta_2, A \multimap B \Vdash C}\ {\multimap}L$$

Admissibility of Cut and Identity

- Fundamental criteria for sensible sequent calculus
  - Cut-free system provides meaning explanation
  - Entails harmony [Dummett'76]
- Define $\Delta \Vdash^* A$ like $\Delta \Vdash A$, without cut, and id_A for atomic A only

Theorem (Admissibility of Cut and Identity)

Cut and identity are admissible in $\Vdash^*$.

$$\frac{\Delta_1 \Vdash^* A \quad \Delta_2, A \Vdash^* C}{\Delta_1, \Delta_2 \Vdash^* C}\ \mathrm{cut}_A \qquad\qquad \frac{}{A \Vdash^* A}\ \mathrm{id}_A$$

Proof.

Cut by nested induction on A and deductions of premises.
Identity by induction on A. □

Cut Elimination

Theorem (Cut and Identity Elimination)

If $\Delta \Vdash A$ *then* $\Delta \Vdash^{*} A$

Proof.

By structural induction on deduction of $\Delta \Vdash A$, using admissibility of cut and identity. $\square$

Correspondence: Axiomatic and Sequent Calculus

Theorem (Soundness of Sequent Calculus)

If $\Delta \Vdash A$ *then* $\Delta \vdash A$*.*

Proof.

By induction on the given deduction. $\square$

Theorem (Completeness of Sequent Calculus)

If $\Delta \vdash A$ *then* $\Delta \Vdash A$*.*

Proof.

By induction on the given deduction. $\square$

The Exponential of Linear Logic

- Tackling the modality $!A$ where $A \to B \simeq {!A} \multimap B$
- Same blueprint
  - Axiomatic formulation
  - Hypothetical Hilbert system
  - Deduction theorem(s)
  - Sequent calculus
  - Cut and identity elimination
  - Correspondence
- Constructing calculi for weaker modalities than $!A$.
  - Fragments are identified by subset of axioms

Axiomatizing the Exponential

- Rule of Necessitation

$$\frac{\vdash A}{\vdash\, !A}\ \text{NEC}$$

- Axioms of S4

	Name	Accessibility	Linear
$\vdash\, !(A \multimap B) \multimap\, !A \multimap\, !B$	(K)	[normal]	[!,?]
$\vdash\, !A \multimap A$	(T)	[reflexivity]	[dereliction]
$\vdash\, !A \multimap\, !!A$	(4)	[transitivity]	[digging]

- Controlled weakening and contraction

$\vdash A \multimap\, !B \multimap A$	(W)	[weakening]
$\vdash (!B \multimap\, !B \multimap A) \multimap (!B \multimap A)$	(C)	[contraction]

Linear K

- Write $\Box A$ for fragments of $!A$
- Linear K: Axiom (K) and necessitation

$$\vdash \Box(A \multimap B) \multimap \Box A \multimap \Box B \quad (K)$$

$$\frac{\vdash A}{\vdash \Box A}\ \text{NEC}$$

- Is there a corresponding sequent calculus?
- Is there a version of the deduction theorem?
- Start with sequent calculus

Validity as a Linear Categorical Judgment

- Judgment Γ valid **;** Δ true $\Vdash A$ true
  - Γ: assumed valid (= true in all reachable worlds)
  - Δ: assumed true in current world
  - A: to prove true in current world
- Judgment Γ valid $\Vdash A$ valid (conceptual)
- All antecedents are linear!
- Judgmental principles: inclusion and independence

$$\left[\ \frac{\Gamma \text{ true} \Vdash A \text{ true}}{\Gamma \text{ valid} \Vdash A \text{ valid}}\ \right]$$

$$\left[\ \frac{\Gamma_1 \text{ valid} \Vdash A \text{ valid} \quad \Gamma_2 \text{ valid}, A \text{ valid} \ \mathbf{;}\ \Delta \text{ true} \Vdash C \text{ true}}{\Gamma_1, \Gamma_2 \text{ valid} \ \mathbf{;}\ \Delta \text{ true} \Vdash C \text{ true}}\ \right]$$

- Truth can depend on validity
- Validity cannot depend on truth

Internalizing Linear K-Validity as $\Box A$

- Use only $\Gamma\,;\Delta \Vdash A$ (eliding "valid" and "true")
- $(\Gamma \text{ valid} \Vdash A \text{ valid}) \simeq (\bullet\,;\Gamma \text{ true} \Vdash A \text{ true})$

$$\frac{}{\bullet\,; A \Vdash A}\,\mathrm{id}_A \qquad \frac{\Gamma_1\,;\Delta_1 \Vdash A \quad \Gamma_2\,;\Delta_2, A \Vdash C}{\Gamma_1, \Gamma_2\,;\Delta_1, \Delta_2 \Vdash C}\,\mathrm{cut}_A$$

$$\frac{\Gamma\,;\Delta, A \Vdash B}{\Gamma\,;\Delta \Vdash A \multimap B}\,{\multimap}R \qquad \frac{\Gamma_1\,;\Delta_1 \Vdash A \quad \Gamma_2\,;\Delta_2, B \Vdash C}{\Gamma_1, \Gamma_2\,;\Delta_1, \Delta_2, A \multimap B \Vdash C}\,{\multimap}L$$

$$\frac{\bullet\,;\Gamma \Vdash A}{\Gamma\,;\bullet \Vdash \Box A}\,\Box R \qquad \frac{\Gamma, A\,;\Delta \Vdash C}{\Gamma\,;\Delta, \Box A \Vdash C}\,\Box L$$

$$\frac{\bullet\,;\Gamma_1 \Vdash A \quad \Gamma_2, A\,;\Delta \Vdash C}{\Gamma_1, \Gamma_2\,;\Delta \Vdash C}\,\mathrm{cut}^{\Box}_A$$

Cut on Validity

- Recall definition

$$\left[\frac{\Gamma\ \mathsf{true} \Vdash A\ \mathsf{true}}{\Gamma\ \mathsf{valid} \Vdash A\ \mathsf{valid}} \right]$$

- Justifies second form of cut

$$\frac{\dfrac{\bullet\ ;\ \Gamma_1 \Vdash A}{[\Gamma_1\ \mathsf{valid} \Vdash A\ \mathsf{valid}]} \quad \Gamma_2, A\ \mathsf{valid}\ ;\ \Delta \Vdash C}{\Gamma_1, \Gamma_2\ \mathsf{valid}\ ;\ \Delta \Vdash C}\ \mathsf{cut}_A^{\Box}$$

Admissibility of Cut

Theorem (Admissibility of Cut)

$$\frac{\Gamma_1 \,;\, \Delta_1 \Vdash^* A \quad \Gamma_2 \,;\, \Delta_2, A \Vdash^* C}{\Gamma_1, \Gamma_2 \,;\, \Delta_1, \Delta_2 \Vdash^* C}\ \mathrm{cut}_A$$

$$\frac{\bullet \,;\, \Gamma_1 \Vdash^* A \quad \Gamma_2, A \,;\, \Delta \Vdash^* C}{\Gamma_1, \Gamma_2 \,;\, \Delta \Vdash^* C}\ \mathrm{cut}_A^{\Box}$$

Proof.

By mutual nested induction on A and the deduction of the two premises. □

Admissibility of Identity

Theorem (Admissibility of Identity)

$$\frac{}{\cdot\ ;\ A \Vdash^* A}\ \mathrm{id}_A$$

Proof.

By induction on the structure of A. Sample case:

Case: $A = \Box A'$.

$$\dfrac{\dfrac{\dfrac{}{\bullet\ ;\ A' \Vdash^* A'}\ i.h.(A')}{A'\ ;\ \bullet \Vdash^* \Box A'}\ \Box_R}{\bullet\ ;\ \Box A' \Vdash^* \Box A'}\ \Box_L$$

Hypothetical Hilbert System

- Construct in analogy with sequent calculus

$$\frac{}{\bullet\,;\, A \vdash A}\ \mathrm{HYP} \qquad \frac{\Gamma_1\,;\,\Delta_1 \vdash A \multimap B \quad \Gamma_2\,;\,\Delta_2 \vdash A}{\Gamma_1, \Gamma_2\,;\,\Delta_1, \Delta_2 \vdash B}\ \mathrm{MP}$$

$$\frac{}{\bullet\,;\,\bullet \vdash \text{axiom}} \qquad \frac{\bullet\,;\,\Gamma \vdash A}{\Gamma\,;\,\bullet \vdash \Box A}\ \mathrm{NEC}$$

- Axioms

$$\begin{array}{ll} \vdash (A \multimap B) \multimap (B \multimap C) \multimap (A \multimap C) & (L) \\ \vdash A \multimap A & (I) \\ \vdash (A \multimap B \multimap C) \multimap (B \multimap A \multimap C) & (X) \\ \vdash \Box(A \multimap B) \multimap \Box A \multimap \Box B & (K) \end{array}$$

Two Deduction Theorems

Theorem (Deduction)

$$\frac{\Gamma \,;\, \Delta, A \vdash B}{\Gamma \,;\, \Delta \vdash A \multimap B}\ \mathrm{DED} \qquad \frac{\Gamma, A \,;\, \Delta \vdash B}{\Gamma \,;\, \Delta \vdash \Box A \multimap B}\ \mathrm{DED}^{\Box}$$

Proof.

By mutual induction on the given deductions. Sample:

Case:

$$\frac{\bullet \,;\, \Gamma, A \vdash B}{\Gamma, A \,;\, \bullet \vdash \Box B}\ \mathrm{NEC}$$

Two Deduction Theorems

Theorem (Deduction)

$$\frac{\Gamma \,;\, \Delta, A \vdash B}{\Gamma \,;\, \Delta \vdash A \multimap B}\ \text{DED} \qquad \frac{\Gamma, A \,;\, \Delta \vdash B}{\Gamma \,;\, \Delta \vdash \Box A \multimap B}\ \text{DED}^{\Box}$$

Proof.

By mutual induction on the given deductions. Sample:

Case:

$$\frac{\bullet \,;\, \Gamma, A \vdash B}{\Gamma, A \,;\, \bullet \vdash \Box B}\ \text{NEC}$$

$$\Gamma \,;\, \bullet \vdash \Box A \multimap \Box B$$

Two Deduction Theorems

Theorem (Deduction)

$$\frac{\Gamma \,;\, \Delta, A \vdash B}{\Gamma \,;\, \Delta \vdash A \multimap B}\ \text{DED} \qquad\qquad \frac{\Gamma, A \,;\, \Delta \vdash B}{\Gamma \,;\, \Delta \vdash \Box A \multimap B}\ \text{DED}^{\Box}$$

Proof.

By mutual induction on the given deductions. Sample:

Case:

$$\frac{\bullet \,;\, \Gamma, A \vdash B}{\Gamma, A \,;\, \bullet \vdash \Box B}\ \text{NEC}$$

$\bullet \,;\, \Gamma \vdash A \multimap B$ i.h.(DED)

$\Gamma \,;\, \bullet \vdash \Box A \multimap \Box B$

Two Deduction Theorems

Theorem (Deduction)

$$\frac{\Gamma \,;\, \Delta, A \vdash B}{\Gamma \,;\, \Delta \vdash A \multimap B}\ \text{DED} \qquad\qquad \frac{\Gamma, A \,;\, \Delta \vdash B}{\Gamma \,;\, \Delta \vdash \Box A \multimap B}\ \text{DED}^{\Box}$$

Proof.

By mutual induction on the given deductions. Sample:

Case:

$$\frac{\bullet \,;\, \Gamma, A \vdash B}{\Gamma, A \,;\, \bullet \vdash \Box B}\ \text{NEC}$$

$$\begin{array}{ll} \bullet \,;\, \Gamma \vdash A \multimap B & \text{i.h.(DED)} \\ \Gamma \,;\, \bullet \vdash \Box(A \multimap B) & \text{NEC} \\ & \\ \Gamma \,;\, \bullet \vdash \Box A \multimap \Box B & \end{array}$$

Two Deduction Theorems

Theorem (Deduction)

$$\frac{\Gamma\,;\,\Delta, A \vdash B}{\Gamma\,;\,\Delta \vdash A \multimap B}\ \text{DED} \qquad \frac{\Gamma, A\,;\,\Delta \vdash B}{\Gamma\,;\,\Delta \vdash \Box A \multimap B}\ \text{DED}^{\Box}$$

Proof.

By mutual induction on the given deductions. Sample:

Case:

$$\frac{\bullet\,;\,\Gamma, A \vdash B}{\Gamma, A\,;\,\bullet \vdash \Box B}\ \text{NEC}$$

$$\begin{array}{lr}
\bullet\,;\,\Gamma \vdash A \multimap B & \text{i.h.(DED)} \\
\Gamma\,;\,\bullet \vdash \Box(A \multimap B) & \text{NEC} \\
\bullet\,;\,\bullet \vdash \Box(A \multimap B) \multimap (\Box A \multimap \Box B) & (K) \\
\Gamma\,;\,\bullet \vdash \Box A \multimap \Box B &
\end{array}$$

Two Deduction Theorems

Theorem (Deduction)

$$\frac{\Gamma \,;\, \Delta, A \vdash B}{\Gamma \,;\, \Delta \vdash A \multimap B}\ \mathrm{DED} \qquad \frac{\Gamma, A \,;\, \Delta \vdash B}{\Gamma \,;\, \Delta \vdash \Box A \multimap B}\ \mathrm{DED}^{\Box}$$

Proof.

By mutual induction on the given deductions. Sample:

Case:

$$\frac{\bullet \,;\, \Gamma, A \vdash B}{\Gamma, A \,;\, \bullet \vdash \Box B}\ \mathrm{NEC}$$

$\bullet \,;\, \Gamma \vdash A \multimap B$	i.h.(DED)
$\Gamma \,;\, \bullet \vdash \Box(A \multimap B)$	NEC
$\bullet \,;\, \bullet \vdash \Box(A \multimap B) \multimap (\Box A \multimap \Box B)$	(K)
$\Gamma \,;\, \bullet \vdash \Box A \multimap \Box B$	MP

Correspondence

Theorem (Correspondence for Linear K)

$\Gamma\,;\,\Delta \vdash A$ *iff* $\Gamma\,;\,\Delta \Vdash A$

Proof.

In each direction by structural induction on given deduction. □

Adding Weakening and Contraction

- Structural axioms for necessity

$$\vdash A \multimap \Box B \multimap A \qquad (W)$$
$$\vdash (\Box A \multimap \Box A \multimap C) \multimap (\Box A \multimap C) \qquad (C)$$

- Deduction theorems as before
- Sequent calculus (A^+ denotes multiple copies of A)

$$\frac{\Gamma \,;\, \Delta \Vdash C}{\Gamma, A \,;\, \Delta \Vdash C}\ \mathrm{wk} \qquad \frac{\Gamma, A, A \,;\, \Delta \Vdash C}{\Gamma, A \,;\, \Delta \Vdash C}\ \mathrm{ct}$$

$$\frac{\bullet \,;\, \Gamma_1 \Vdash A \quad \Gamma_2, A^+ \,;\, \Delta \Vdash C}{\Gamma_1, \Gamma_2 \,;\, \Delta \Vdash C}\ \mathrm{cut}_A^{\Box +}$$

- Other formulations are possible

Elementary Linear Logic

- Linear KWC is elementary linear logic
  - Captures elementary recursive functions [Danos & Joinet'01]
- KTWC adds $\vdash \Box A \multimap A$
  - Can represent all recursive functions [D&J remark]
- K4WC adds $\vdash \Box A \multimap \Box\Box A$
- K4TWC is intuitionistic linear logic
- Linear KT[WC] and K4[WC] have judgmental formulations (next)

Reflexivity (= Dereliction)

- Axiomatically:

$$\vdash \Box A \multimap A \quad (T)$$

- Sequent calculus:

$$\frac{\Gamma \,\mathbf{;}\, \Delta, A \Vdash C}{\Gamma, A \,\mathbf{;}\, \Delta \Vdash C}\ \text{refl}$$

- All metatheorems carry over, including correspondence.

Transitivity (= Digging)

- Axiomatically:

$$\vdash \Box A \multimap \Box\Box A \quad (4)$$

- Sequent calculus:

$$\frac{\Gamma_1 \,;\, \Gamma_2 \Vdash A}{\Gamma_1, \Gamma_2 \,;\, \bullet \Vdash \Box A} \,\Box R$$

$$\frac{\Gamma_1' \,;\, \Gamma_1'' \Vdash A \qquad \Gamma_2, A \,;\, \Delta \Vdash C}{\Gamma_1', \Gamma_1'', \Gamma_2 \,;\, \Delta \Vdash C} \,\mathrm{cut}^{\Box}$$

- All metatheorems carry over

Story So Far

- Base axiomatic system

$$
\begin{array}{ll}
\vdash (A \multimap B) \multimap (B \multimap C) \multimap (A \multimap C) & (L) \\
\vdash A \multimap A & (I) \\
\vdash (A \multimap B \multimap C) \multimap (B \multimap A \multimap C) & (X) \\
\vdash \Box(A \multimap B) \multimap (\Box A \multimap \Box B) & (K)
\end{array}
$$

$$
\frac{\vdash A \multimap B \quad \vdash A}{\vdash B}\ \mathrm{MP} \qquad \frac{\vdash A}{\vdash \Box A}\ \mathrm{NEC}
$$

- Additional axioms

$$
\begin{array}{lll}
\vdash \Box A \multimap A & (T) & \text{dereliction} \\
\vdash \Box A \multimap \Box\Box A & (4) & \text{digging} \\
\vdash C \multimap \Box A \multimap C & (W) & \text{weakening} \\
\vdash (\Box A \multimap \Box A \multimap C) \multimap (\Box A \multimap C) & (C) & \text{contraction}
\end{array}
$$

- All* combinations yield defensible logics with structural cut elimination and identity.

Techniques

- Separating judgments and propositions
  - Validity derived via inclusion and independence
  - Truth can depend on validity
  - Validity cannot depend on truth
- Sequent calculus with validity and truth
  - Valid and true antecedents
  - Additional judgmental rules for T, 4, W, C
- Hypothetical Hilbert system as bridge
  - Validity judgment as additional hypotheses
  - Two deduction theorems
- Various combinations can be "optimized"

Another Decomposition: Adjoint Logic

- Combine intuitionistic and intuitionistic linear logic via an adjunction [Benton'94]
  - Two functors F and G, F left adjoint to G
  - Syntax as modal operators $G\,A$ and $F\,X$
  - Decompose $!A \simeq F\,(G\,A)$
- Generalized to multi-modal logics [Reed'09]
- Applies to polarization [Laurent'99] [Pf. & Griffith'15]
- Question: Does it apply to weaker logics?

Adjoint Logic

- Two-level system [Benton'94] ($\downarrow = F, \uparrow = G$)

$$\begin{array}{llll} \text{Unrestricted} & A_{\mathsf{U}} & ::= & A_{\mathsf{U}} \to A_{\mathsf{U}} \mid \uparrow A_{\mathsf{L}} \\ \text{Linear} & A_{\mathsf{L}} & ::= & A_{\mathsf{L}} \multimap A_{\mathsf{L}} \mid \downarrow A_{\mathsf{U}} \end{array}$$

- Represent $!A_{\mathsf{L}} \simeq \downarrow\uparrow A_{\mathsf{L}}$
- Now both levels are linear
  - No weakening or contraction
  - No analogue of dereliction or digging
  - Read: U = Upper level, L = Lower level
  - Upper level represents validity
  - Lower level represents truth

Adjoint K, Judgmental Rules

- $\Gamma ::= \bullet \mid \Gamma, A_{\mathsf{U}}$
- $\Delta ::= \bullet \mid \Delta, A_{\mathsf{L}}$
- Judgments $\Gamma \Vdash A_{\mathsf{U}}$ and $\Gamma \,;\, \Delta \Vdash A_{\mathsf{L}}$

$$\frac{}{A_{\mathsf{U}} \Vdash A_{\mathsf{U}}}\ \mathsf{id}_{\mathsf{U}} \qquad \frac{}{\bullet \,;\, A_{\mathsf{L}} \Vdash A_{\mathsf{L}}}\ \mathsf{id}_{\mathsf{L}}$$

$$\frac{\Gamma_1 \Vdash A_{\mathsf{U}} \quad \Gamma_2, A_{\mathsf{U}} \Vdash C_{\mathsf{U}}}{\Gamma_1, \Gamma_2 \Vdash C_{\mathsf{U}}}\ \mathsf{cut}_{\mathsf{UU}} \qquad \frac{\Gamma_1 \,;\, \Delta_1 \Vdash A_{\mathsf{L}} \quad \Gamma_2 \,;\, \Delta_2, A_{\mathsf{L}} \Vdash C_{\mathsf{L}}}{\Gamma_1, \Gamma_2 \,;\, \Delta_1, \Delta_2 \Vdash C_{\mathsf{L}}}\ \mathsf{cut}_{\mathsf{LL}}$$

$$\frac{\Gamma_1 \Vdash A_{\mathsf{U}} \quad \Gamma_2, A_{\mathsf{U}} \,;\, \Delta \Vdash C_{\mathsf{L}}}{\Gamma_1, \Gamma_2 \,;\, \Delta \Vdash C_{\mathsf{L}}}\ \mathsf{cut}_{\mathsf{UL}}$$

Adjoint K, Modal Rules

$$\frac{\Gamma \Vdash A_{\mathsf{U}}}{\Gamma \,;\, \bullet \Vdash {\downarrow} A_{\mathsf{U}}}\;{\downarrow}R \qquad \frac{\Gamma, A_{\mathsf{U}} \,;\, \Delta \Vdash C_{\mathsf{L}}}{\Gamma \,;\, \Delta, {\downarrow} A_{\mathsf{U}} \Vdash C_{\mathsf{L}}}\;{\downarrow}L$$

$$\frac{\Gamma \,;\, \bullet \Vdash A_{\mathsf{L}}}{\Gamma \Vdash {\uparrow} A_{\mathsf{L}}}\;{\uparrow}R \qquad \frac{\Gamma \,;\, \Delta, A_{\mathsf{L}} \Vdash C_{\mathsf{L}}}{\Gamma, {\uparrow} A_{\mathsf{L}} \,;\, \Delta \Vdash C_{\mathsf{L}}}\;{\uparrow}L$$

Adjoint K, Axiomatic System

- Two judgments $\vdash^{\mathsf{L}} A_{\mathsf{L}}$ and $\vdash^{\mathsf{U}} A_{\mathsf{U}}$
- Implicational fragments (eliding level annotations)

$$
\begin{array}{ll}
\vdash^{\mathsf{L}} (A \multimap B) \multimap (B \multimap C) \multimap (A \multimap C) & (L)_{\mathsf{L}} \\
\vdash^{\mathsf{L}} A \multimap A & (I)_{\mathsf{L}} \\
\vdash^{\mathsf{L}} (A \multimap B \multimap C) \multimap (B \multimap A \multimap C) & (X)_{\mathsf{L}}
\end{array}
$$

$$
\begin{array}{ll}
\vdash^{\mathsf{U}} (A \to B) \to (B \to C) \to (A \to C) & (L)_{\mathsf{U}} \\
\vdash^{\mathsf{U}} A \to A & (I)_{\mathsf{U}} \\
\vdash^{\mathsf{U}} (A \to B \to C) \to (B \to A \to C) & (X)_{\mathsf{U}}
\end{array}
$$

$$
\frac{\vdash^{\mathsf{L}} A \multimap B \quad \vdash^{\mathsf{L}} A}{\vdash^{\mathsf{L}} B}\ \mathrm{MP}_{\mathsf{U}}
\qquad\qquad
\frac{\vdash^{\mathsf{U}} A \to B \quad \vdash^{\mathsf{U}} A}{\vdash^{\mathsf{U}} B}\ \mathrm{MP}_{\mathsf{L}}
$$

Adjoint K, Modal Operators

- Mixed analogues of K

$$\overset{\mathsf{L}}{\vdash} \downarrow(A_{\mathsf{U}} \to B_{\mathsf{U}}) \multimap (\downarrow A_{\mathsf{U}} \multimap \downarrow B_{\mathsf{U}}) \quad (K)_{\mathsf{L}}$$
$$\overset{\mathsf{U}}{\vdash} \uparrow(A_{\mathsf{L}} \multimap B_{\mathsf{L}}) \to (\uparrow A_{\mathsf{L}} \to \uparrow B_{\mathsf{L}}) \quad (K)_{\mathsf{U}}$$

- Mixed analogues of NEC

$$\frac{\overset{\mathsf{U}}{\vdash} A_{\mathsf{U}}}{\overset{\mathsf{L}}{\vdash} \downarrow A_{\mathsf{U}}} \downarrow \qquad \frac{\overset{\mathsf{L}}{\vdash} A_{\mathsf{L}}}{\overset{\mathsf{U}}{\vdash} \uparrow A_{\mathsf{L}}} \uparrow$$

- Adjunction properties

$$\overset{\mathsf{L}}{\vdash} \downarrow\uparrow A_{\mathsf{L}} \multimap A_{\mathsf{L}} \quad (J)_{\mathsf{L}}$$
$$\overset{\mathsf{L}}{\vdash} A_{\mathsf{U}} \to \uparrow\downarrow A_{\mathsf{U}} \quad (J)_{\mathsf{U}}$$

Adjoint K, Hypothetical Hilbert System

- Judgments $\Gamma \vdash^{\mathsf{U}} A_{\mathsf{U}}$ and $\Gamma \,;\, \Delta \vdash^{\mathsf{L}} A_{\mathsf{L}}$
- In analogy with sequent calculus. For example:

$$\frac{\Gamma \vdash^{\mathsf{U}} A_{\mathsf{U}}}{\Gamma \,;\, \bullet \vdash^{\mathsf{L}} {\downarrow} A_{\mathsf{U}}} \downarrow \qquad \frac{\Gamma \,;\, \bullet \vdash^{\mathsf{L}} A_{\mathsf{L}}}{\Gamma \vdash^{\mathsf{U}} {\uparrow} A_{\mathsf{L}}} \uparrow$$

- Three deduction theorems

$$\frac{\Gamma, A_{\mathsf{U}} \vdash^{\mathsf{U}} B_{\mathsf{U}}}{\Gamma \vdash^{\mathsf{U}} A_{\mathsf{U}} \to B_{\mathsf{U}}} \ \mathrm{DED}_{\mathsf{UU}} \qquad \frac{\Gamma \,;\, \Delta, A_{\mathsf{L}} \vdash^{\mathsf{L}} B_{\mathsf{L}}}{\Gamma \,;\, \Delta \vdash^{\mathsf{L}} A_{\mathsf{L}} \multimap B_{\mathsf{L}}} \ \mathrm{DED}_{\mathsf{LL}}$$

$$\frac{\Gamma, A_{\mathsf{U}} \,;\, \Delta \vdash^{\mathsf{L}} B_{\mathsf{L}}}{\Gamma \,;\, \Delta \vdash^{\mathsf{L}} {\downarrow} A_{\mathsf{U}} \multimap B_{\mathsf{L}}} \ \mathrm{DED}_{\mathsf{UL}}$$

Adjunct K, Correspondence

Theorem (Correspondence for Adjunct K)

(i) $\Gamma\,;\Delta \Vdash A_{\mathsf{L}}$ *iff* $\Gamma\,;\Delta \vdash^{\mathsf{L}} A_{\mathsf{L}}$
(ii) $\Gamma \Vdash A_{\mathsf{U}}$ *iff* $\Gamma \vdash^{\mathsf{U}} A_{\mathsf{U}}$

- Weakening and contraction for U are orthogonal
- Under $!A_{\mathsf{L}} \simeq {\downarrow}{\uparrow}A_{\mathsf{L}}$
  - $!A_{\mathsf{L}} \multimap A_{\mathsf{L}}$ follows by $(J)_{\mathsf{L}}$
  - $!A_{\mathsf{L}} \multimap\, !!A_{\mathsf{L}}$ follows by $(J)_{\mathsf{U}}$
- Unavoidable? Linear K violates stratification of syntax:

$$\frac{\bullet\,;\Gamma \Vdash A}{\Gamma\,;\bullet \Vdash \Box A}\;\Box R$$

Summary

- Four properties of $!A$, reflexivity (T), transitivity (4), weakening (W), contraction (C), can be mixed and matched
  - Axiomatic system by subsetting axioms
  - Sequent systems via judgmental distinctions and rules
  - Structural cut elimination and identity for all* systems
  - Clean meaning explanations
  - Yields elementary linear logic ($=$ KWC)
  - Applications for other systems?
- Adjoint decomposition $!A \simeq {\downarrow}{\uparrow}A$
  - Weakening and contraction orthogonal
  - Reflexivity and transitivity appear inevitable

Further Observations and Questions

- Conjecture generalization to a pre-order of levels
  - For independence and inclusion [Pientka]
  - For adjoint approach [Reed'09]
  - See also subexponentials [Nigam & Miller'09]
  - Applications in session types [Pf & Griffith'15]
- Compatible with constructive possibility [Pf'13]
- Not fully compatible with world-indexed truth [Simpson'94]
  - Violates independence $(\Diamond A \multimap \Box B) \multimap \Box(A \multimap B)$
  - Related discrepancies for $A \oplus B$, $\mathbf{0}$
  - Recover via tethering? [Pf'13]
- Can we construct fragmentary dependent type theories?
- Are there further structural complexity classes?
- Other applications?

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- Rowan Davies, Rob Simmons, Henry DeYoung, . . .
- Details in unpublished "Weather Report" [Pf'13]

Teaser: Simpson's Axiom

$$\Vdash^* (\Diamond A \multimap \Box B) \multimap \Box(A \multimap B)$$

Teaser: Simpson's Axiom

$$\begin{array}{ll} \Diamond A \multimap \Box B \Vdash^{*} \Box(A \multimap B) & \\ \Vdash^{*} (\Diamond A \multimap \Box B) \multimap \Box(A \multimap B) & \multimap R \end{array}$$

Teaser: Simpson's Axiom

No rule applies!

$\Diamond A \multimap \Box B \Vdash^* \Box(A \multimap B)$

$\Vdash^* (\Diamond A \multimap \Box B) \multimap \Box(A \multimap B) \qquad \multimap R$

Teaser: Simpson's Axiom

No rule applies!

$\Diamond A \multimap \Box B \Vdash^* \Box(A \multimap B)$

$\Vdash^* (\Diamond A \multimap \Box B) \multimap \Box(A \multimap B) \qquad \multimap R$

- Not provable in any presented system
- Proof would violate independence!

Teaser: World-Indexed Truth

- $A[w]$ means A true in world w
- $w_0 \leq w_1$ means w_1 is accessible from w_0

$$\Vdash ((\Diamond A \multimap \Box B) \multimap \Box(A \multimap B))[w_0]$$

Teaser: World-Indexed Truth

- $A[w]$ means A true in world w
- $w_0 \leq w_1$ means w_1 is accessible from w_0

$$\dfrac{(\Diamond A \multimap \Box B)[w_0] \Vdash \Box(A \multimap B)[w_0]}{\Vdash ((\Diamond A \multimap \Box B) \multimap \Box(A \multimap B))[w_0]} \multimap R$$

Teaser: World-Indexed Truth

- $A[w]$ means A true in world w
- $w_0 \leq w_1$ means w_1 is accessible from w_0

$$
\begin{array}{ll}
(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1 \Vdash (A \multimap B)[w_1] & \\
(\Diamond A \multimap \Box B)[w_0] \Vdash \Box(A \multimap B)[w_0] & \Box R \\
\Vdash ((\Diamond A \multimap \Box B) \multimap \Box(A \multimap B))[w_0] & \multimap R
\end{array}
$$

Teaser: World-Indexed Truth

- $A[w]$ means A true in world w
- $w_0 \leq w_1$ means w_1 is accessible from w_0

$$
\begin{array}{lr}
(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1, A[w_1] \Vdash B[w_1] & \\
(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1 \Vdash (A \multimap B)[w_1] & \multimap R \\
(\Diamond A \multimap \Box B)[w_0] \Vdash \Box(A \multimap B)[w_0] & \Box R \\
\Vdash ((\Diamond A \multimap \Box B) \multimap \Box(A \multimap B))[w_0] & \multimap R
\end{array}
$$

Teaser: World-Indexed Truth

- $A[w]$ means A true in world w
- $w_0 \leq w_1$ means w_1 is accessible from w_0

$$w_0 \leq w_1, A[w_1] \Vdash \Diamond A[w_0]$$

$$\Box B[w_0], w_0 \leq w_1 \Vdash B[w_1]$$

$$\begin{array}{lr}
(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1, A[w_1] \Vdash B[w_1] & \multimap L \\
(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1 \Vdash (A \multimap B)[w_1] & \multimap R \\
(\Diamond A \multimap \Box B)[w_0] \Vdash \Box(A \multimap B)[w_0] & \Box R \\
\Vdash ((\Diamond A \multimap \Box B) \multimap \Box(A \multimap B))[w_0] & \multimap R
\end{array}$$

Teaser: World-Indexed Truth

- $A[w]$ means A true in world w
- $w_0 \leq w_1$ means w_1 is accessible from w_0

$$w_0 \leq w_1, A[w_1] \Vdash \Diamond A[w_0]$$

$$\begin{array}{ll}
B[w_1] \Vdash B[w_1] & \\
\Box B[w_0], w_0 \leq w_1 \Vdash B[w_1] & \Box L,\ (w_0 \leq w_1) \\
 & \\
(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1, A[w_1] \Vdash B[w_1] & \multimap L \\
(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1 \Vdash (A \multimap B)[w_1] & \multimap R \\
(\Diamond A \multimap \Box B)[w_0] \Vdash \Box(A \multimap B)[w_0] & \Box R \\
\Vdash ((\Diamond A \multimap \Box B) \multimap \Box(A \multimap B))[w_0] & \multimap R
\end{array}$$

Teaser: World-Indexed Truth

- $A[w]$ means A true in world w
- $w_0 \leq w_1$ means w_1 is accessible from w_0

$$w_0 \leq w_1, A[w_1] \Vdash \Diamond A[w_0]$$

$$\begin{array}{lr}
B[w_1] \Vdash B[w_1] & \mathrm{id}_B \\
\Box B[w_0], w_0 \leq w_1 \Vdash B[w_1] & \Box L,\ (w_0 \leq w_1) \\
 & \\
(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1, A[w_1] \Vdash B[w_1] & \multimap L \\
(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1 \Vdash (A \multimap B)[w_1] & \multimap R \\
(\Diamond A \multimap \Box B)[w_0] \Vdash \Box(A \multimap B)[w_0] & \Box R \\
\Vdash ((\Diamond A \multimap \Box B) \multimap \Box(A \multimap B))[w_0] & \multimap R
\end{array}$$

Teaser: World-Indexed Truth

- $A[w]$ means A true in world w
- $w_0 \leq w_1$ means w_1 is accessible from w_0

$$w_0 \leq w_1, A[w_1] \Vdash A[w_1]$$
$$w_0 \leq w_1, A[w_1] \Vdash \Diamond A[w_0] \qquad \Diamond R,\ (w_0 \leq w_1)$$

$$B[w_1] \Vdash B[w_1] \qquad \mathrm{id}_B$$
$$\Box B[w_0], w_0 \leq w_1 \Vdash B[w_1] \qquad \Box L,\ (w_0 \leq w_1)$$

$$(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1, A[w_1] \Vdash B[w_1] \qquad \multimap L$$
$$(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1 \Vdash (A \multimap B)[w_1] \qquad \multimap R$$
$$(\Diamond A \multimap \Box B)[w_0] \Vdash \Box(A \multimap B)[w_0] \qquad \Box R$$
$$\Vdash ((\Diamond A \multimap \Box B) \multimap \Box(A \multimap B))[w_0] \qquad \multimap R$$

Teaser: World-Indexed Truth

- $A[w]$ means A true in world w
- $w_0 \leq w_1$ means w_1 is accessible from w_0

$$w_0 \leq w_1, A[w_1] \Vdash A[w_1] \qquad \mathrm{id}_A$$
$$w_0 \leq w_1, A[w_1] \Vdash \Diamond A[w_0] \qquad \Diamond R,\ (w_0 \leq w_1)$$

$$B[w_1] \Vdash B[w_1] \qquad \mathrm{id}_B$$
$$\Box B[w_0], w_0 \leq w_1 \Vdash B[w_1] \qquad \Box L,\ (w_0 \leq w_1)$$

$$(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1, A[w_1] \Vdash B[w_1] \qquad \multimap L$$
$$(\Diamond A \multimap \Box B)[w_0], w_0 \leq w_1 \Vdash (A \multimap B)[w_1] \qquad \multimap R$$
$$(\Diamond A \multimap \Box B)[w_0] \Vdash \Box(A \multimap B)[w_0] \qquad \Box R$$
$$\Vdash ((\Diamond A \multimap \Box B) \multimap \Box(A \multimap B))[w_0] \qquad \multimap R$$

- Provable without any assumption on accessibility!