Herbrand-confluence in the classical sequent calculus

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Introduction

- Confluence in proof theory
- Cut-elimination theorem [Gentzen '34]
- Cut-elimination as proof rewriting
- Not confluent
- *Which confluence results can be obtained?*
- Herbrand-confluence

Outline

- ⇒ Classical sequent calculus
- Non-confluence
- Confluent restrictions
- Herbrand-confluence
- Proofs and grammars

The sequent calculus (1/2)

- Formulas: $\wedge, \vee, \neg, \rightarrow, \forall, \exists$
- Sequent: $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_k$
  as formula: $\bigwedge_{i=1}^{n} A_i \rightarrow \bigvee_{j=1}^{k} B_j$
- Propositional rules, e.g.

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Pi \Rightarrow \Lambda, B}{\Gamma, \Pi \Rightarrow \Delta, \Lambda, A \wedge B}\ \wedge_r \qquad \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ \wedge_l$$

- Quantifier rules, e.g.

$$\frac{\Gamma \Rightarrow \Delta, A(\alpha)}{\Gamma \Rightarrow \Delta, \forall x\, A(x)}\ \forall_r \qquad \frac{A(t), \Gamma \Rightarrow \Delta}{\forall x\, A(x), \Gamma \Rightarrow \Delta}\ \forall_l$$

  where α does not occur free in $\Gamma \Rightarrow \Delta, \forall x\, A(x)$.

The sequent calculus (2/2)

- Structural rules: contraction

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathsf{c_l} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \mathsf{c_r}$$

- Structural rules: weakening

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathsf{w_l} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ \mathsf{w_r}$$

- The cut

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

- **Definition.** An **LK**-proof is a tree built from these inference rules and inital sequents of the form $A \Rightarrow A$.
- **Theorem (Sound- & Completeness).** A sequent is provable in **LK** iff it is valid.

An example proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(\alpha,\beta) \Rightarrow P(\alpha,\beta)}{\Rightarrow P(\alpha,\beta), \neg P(\alpha,\beta)}\ \neg_r}{\Rightarrow P(\alpha,\beta), \exists y\, \neg P(\alpha,y)}\ \exists_r}{\Rightarrow \forall y\, P(\alpha,y), \exists y\, \neg P(\alpha,y)}\ \forall_r \quad Q(\alpha) \Rightarrow Q(\alpha)}{\forall y\, P(\alpha,y) \to Q(\alpha) \Rightarrow \exists y\, \neg P(\alpha,y), Q(\alpha)}\ \to_l}{\forall x\, (\forall y\, P(x,y) \to Q(x)) \Rightarrow \exists y\, \neg P(\alpha,y), Q(\alpha)}\ \forall_l}{\forall x\, (\forall y\, P(x,y) \to Q(x)) \Rightarrow \exists y\, \neg P(\alpha,y) \vee Q(\alpha)}\ \vee_r}{\forall x\, (\forall y\, P(x,y) \to Q(x)) \Rightarrow \forall x\, (\exists y\, \neg P(x,y) \vee Q(x))}\ \forall_r
$$

Cut-elimination as proof rewriting (1/3)

Theorem [Gentzen '34]. If a sequent is provable in **LK**, then it is provable in **LK** without cut.

Proof. By local proof reductions:

- Reduction of connectives, e.g.

$$\dfrac{\dfrac{\overset{(\pi_1)}{\Gamma \Rightarrow \Delta, A(\alpha)}}{\Gamma \Rightarrow \Delta, \forall x\, A(x)}\,\forall_r \qquad \dfrac{\overset{(\pi_2)}{A(t), \Pi \Rightarrow \Lambda}}{\forall x\, A(x), \Pi \Rightarrow \Lambda}\,\forall_l}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\text{cut} \quad \to \quad \dfrac{\overset{(\pi_1[\alpha \backslash t])}{\Gamma \Rightarrow \Delta, A(t)} \qquad \overset{(\pi_2)}{A(t), \Pi \Rightarrow \Lambda}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\text{cut}$$

- Rule permutations, e.g.

$$\dfrac{\dfrac{\overset{(\pi_1)}{A, B, \Gamma \Rightarrow \Delta, C}}{A \wedge B, \Gamma \Rightarrow \Delta, C}\,\wedge_l \qquad \overset{(\pi_2)}{C, \Pi \Rightarrow \Lambda}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\text{cut} \quad \to \quad \dfrac{\dfrac{\overset{(\pi_1)}{A, B, \Gamma \Rightarrow \Delta, C} \qquad \overset{(\pi_2)}{C, \Pi \Rightarrow \Lambda}}{A, B, \Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\text{cut}}{A \wedge B, \Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\wedge_l$$

- Reduction of contraction, e.g.

$$
\dfrac{\begin{array}{c}(\pi_1)\\ \Gamma \Rightarrow \Delta, A\end{array} \qquad \dfrac{\begin{array}{c}(\pi_2)\\ A, A, \Pi \Rightarrow \Lambda\end{array}}{A, \Pi \Rightarrow \Lambda}\,\mathsf{c_l}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\mathsf{cut}
$$

$$
\rightarrow \qquad \dfrac{\dfrac{\begin{array}{c}(\pi_1')\\ \Gamma \Rightarrow \Delta, A\end{array} \qquad \dfrac{\begin{array}{c}(\pi_1'')\\ \Gamma \Rightarrow \Delta, A\end{array} \quad \begin{array}{c}(\pi_2)\\ A, A, \Pi \Rightarrow \Lambda\end{array}}{A, \Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\mathsf{cut}}{\Gamma, \Gamma, \Pi \Rightarrow \Delta, \Delta, \Lambda}\,\mathsf{cut}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\mathsf{c}^*
$$

Cut-elimination as proof rewriting (3/3)

- Reduction of weakening, e.g.

$$\dfrac{\begin{matrix}(\pi_1)\\ \Gamma \Rightarrow \Delta, A\end{matrix} \quad \dfrac{\begin{matrix}(\pi_2)\\ \Pi \Rightarrow \Lambda\end{matrix}}{A, \Pi \Rightarrow \Lambda}\,\mathsf{w_l}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\mathsf{cut} \quad \rightarrow \quad \dfrac{\begin{matrix}(\pi_2)\\ \Pi \Rightarrow \Lambda\end{matrix}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\mathsf{w}^*$$

- Reduction of initial sequents, e.g.

$$\dfrac{A \Rightarrow A \quad \begin{matrix}(\pi_1)\\ A, \Gamma \Rightarrow \Delta\end{matrix}}{A, \Gamma \Rightarrow \Delta}\,\mathsf{cut} \quad \rightarrow \quad \begin{matrix}(\pi_1)\\ A, \Gamma \Rightarrow \Delta\end{matrix}$$

Properties of cut-elimination

- $(\mathbf{LK}, \to)$ proof rewriting system
- Gentzen's theorem: weak normalisation (WN)
- Not strongly normalising (SN):

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A \vee A \Rightarrow A, A}\,\vee_{\mathsf{l}}}{A \vee A \Rightarrow A}\,\mathsf{c}_{\mathsf{r}} \qquad \dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A, A \Rightarrow A \wedge A}\,\wedge_{\mathsf{r}}}{A \Rightarrow A \wedge A}\,\mathsf{c}_{\mathsf{l}}}{A \vee A \Rightarrow A \wedge A}\,\mathsf{cut}$$

  has infinite reduction (and inf. many normal forms)
- Many critical pairs

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Lafont's example

- [Girard, Taylor, Lafont '89]: non-confluence of **LK**

$$\dfrac{\dfrac{\dfrac{(\pi_1)}{\Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta, A}\,\mathsf{w_r} \qquad \dfrac{\dfrac{(\pi_2)}{\Pi \Rightarrow \Lambda}}{A, \Pi \Rightarrow \Lambda}\,\mathsf{w_l}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\mathsf{cut}$$

$$\swarrow \qquad\qquad \searrow$$

$$\dfrac{\dfrac{(\pi_1)}{\Gamma \Rightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\mathsf{w}^* \qquad\qquad \dfrac{\dfrac{(\pi_2)}{\Pi \Rightarrow \Lambda}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\,\mathsf{w}^*$$

- In fact: much stronger non-confluence is possible

Complexity of cut-elimination

Theorem [Statman '79, Orevkov '79]. There is a sequence of proofs $(\pi_n : S_n)_{n \geq 1}$ with $|\pi_n|$ polynomial in n s.t. the shortest cut-free proof of S_n has length $\geq 2_n$.

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Proof Sketch [Pudlák, 98].

- $S_n := \mathcal{A}, I(0), \forall x\,(I(x) \to I(s(x))) \Rightarrow I(\mathrm{tt}_n)$
  where $\mathrm{tt}_0 = 2^0$ and $\mathrm{tt}_{n+1} = 2^{\mathrm{tt}_n}$ are terms linear in n
- Construct π_n explicitely, observe $|\pi_n|$ is polynomial
- All cut-free proofs of $\mathcal{A}, I(0), \forall x\,(I(x) \to I(s(x))) \Rightarrow I(\mathrm{tt}_n)$ contain 2_n instances of $\forall x\,(I(x) \to I(s(x)))$.

Non-Confluence in First-Order Logic

Theorem [Baaz, H '11]. There is a sequence of proofs $(\chi_n)_{n\geq 1}$ with $|\chi_n|$ polynomial in n s.t. the number of different normal forms of χ_n is 2_n. Moreover, these normal forms are different in a strong sense (in particular: different Herbrand-sequents).

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Proof Sketch. Start from Pudlák's $(\pi_n)_{n\geq 1}$

- Replace $I(x)$ "number ... constructible"
  by $F(x)$ "term with depth ... constructible"
- Replace axioms $I(0)$ and $(\forall x)(I(x) \rightarrow I(s(x)))$ by non-deterministic term constructors τ^0 and τ^s

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Restricting the reduction

- ($\mathbf{LK}, \to$) is WN, not SN, not confluent – find $\to' \subseteq \to$ s.t.
  - Normal forms of $\to'$ are cut-free proofs
  - $\to'$ is confluent
  - $\to'$ is SN
- Fixing strategy vs. fixing reduction rules
- $\mathbf{LK}^{tq}$ [Danos, Joinet, Schellinx '97]
- Colors left $\overleftarrow{A}$ and right $\overrightarrow{A}$ for subformulas, e.g.

$$\dfrac{\dfrac{(\pi_1)}{\Gamma \Rightarrow \Delta, \overrightarrow{\overleftarrow{A} \wedge \overrightarrow{B}}} \qquad \dfrac{(\pi_2)}{\overleftarrow{\overleftarrow{A} \wedge \overrightarrow{B}}, \Pi \Rightarrow \Lambda}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \text{ cut}$$

- Color subformula occurrences consistently

Properties of $\mathbf{LK}^{tq}$

- Each annotation: SN and almost confluent
- Proof by translation(s) to linear logic
- Covers many existing systems
- Number of annotations vs. number of normal forms

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Herbrand-sequents

- This talk: Σ_1-sequents
$$\forall\overline{x_1}\, A_1, \ldots, \forall\overline{x_n}\, A_n \Rightarrow \exists\overline{x_{n+1}}\, A_{n+1}, \ldots, \exists\overline{x_m}\, A_m$$
with A_i quantifier-free.
- **Definition.** Let $F = Q\overline{x}\, A$ for $Q \in \{\forall, \exists\}$ and A quantifier-free, then a formula $A[\overline{x}\backslash\overline{t}]$ is called *instance* of F.
- **Definition.** Let $\Gamma \Rightarrow \Delta$ be a Σ_1-sequent. A sequent $\Gamma' \Rightarrow \Delta'$ is called *Herbrand-sequent of* $\Gamma \Rightarrow \Delta$ if
  - Every $A' \in \Gamma'(\Delta')$ is instance of some $A \in \Gamma(\Delta)$.
  - $\Gamma' \Rightarrow \Delta'$ is a tautology.
- A cut-free proof π induces a Herbrand-sequent $\mathrm{H}(\pi)$.
(collect all instances from π)
- Generalisation to non-prenex/mixed quantifiers
higher-order logic: expansion trees [Miller '87]

Herbrand-confluence

- **Definition** [Terese '03] **.** Given ARS $(X, \to)$ and an equivalence relation $\approx$ on X the reduction $\to$ is *confluent up to* $\approx$ if for all $x \to y_1$ and $x \to y_2$ there are z_1, z_2 s.t. $y_1 \to z_1$, $y_2 \to z_2$ and $z_1 \approx z_2$.
- "Having the same Herbrand-sequent"
  equivalence relation on cut-free proofs
- **Definition.** A reduction of **LK**-proofs is called *Herbrand-confluent* if it is confluent up to having the same Herbrand-sequent.

Two obstacles (1/2)

- Double-weakening (Lafont's example)
- **Definition.** Let $\rightarrow^{\mathsf{ne}}$ be $\rightarrow$ without reduction of weakening.
- Normal forms $\rightarrow^{\mathsf{ne}}$ are analytic proofs, i.e.,
  H(π) is Herbrand-sequent
- H(π) can be redundant in obvious way
- $\rightarrow^{\mathsf{ne}}$ still has infinitely many normal forms
  (cf. double-contraction)

Two obstacles (2/2)

- Useless contractions on strong quantifiers, e.g.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}(\pi_1)\\ \Gamma \Rightarrow \Delta, A(\alpha), A(\beta)\end{array}}{\Gamma \Rightarrow \Delta, A(\alpha), \forall x\, A(x)}\ \forall_r}{\Gamma \Rightarrow \Delta, \forall x\, A(x), \forall x\, A(x)}\ \forall_r}{\Gamma \Rightarrow \Delta, \forall x\, A(x)}\ c_r$$

- **Definition** A proof is called *pruned* if every $\forall_r$ and $\exists_l$ is applied as far down as possible.
- **Lemma.** For every proof π of $\Gamma \Rightarrow \Delta$ there is a pruned proof π' of $\Gamma \Rightarrow \Delta$ with $|\pi'| \leq |\pi|$.

$$\dfrac{\dfrac{\begin{array}{c}(\pi_1[\beta \backslash \alpha])\\ \Gamma \Rightarrow \Delta, A(\alpha), A(\alpha)\end{array}}{\Gamma \Rightarrow \Delta, A(\alpha)}\ c_r}{\Gamma \Rightarrow \Delta, \forall x\, A(x)}\ \forall_r$$

Results on Herbrand-confluence

- **Definition.** A cut is called Σ_1 (Σ_2) if its cut-formula is of the form $\exists x\, A$ ($\exists x \forall y A$) with A quantifier-free.

Results on Herbrand-confluence

- **Definition.** A cut is called Σ_1 (Σ_2) if its cut-formula is of the form $\exists x\, A$ ($\exists x \forall y A$) with A quantifier-free.
- **Theorem** [H, Straßburger '12] $\to^{\mathsf{ne}}$ on pruned proofs with at most Σ_1-cuts is Herbrand-confluent.
- **Theorem** [Afshari, H, Leigh '15] $\to^{\mathsf{ne}}$ on pruned proofs with at most Σ_2-cuts is Herbrand-confluent.

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Herbrand-sequents and tree languages

- **Def.** A tree language is a set of var.-free first-order terms.
- A Herbrand sequent of

  $$\forall \overline{x_1}\, A_1, \ldots, \forall \overline{x_n}\, A_n \Rightarrow \exists \overline{x_{n+1}}\, A_{n+1}, \ldots, \exists \overline{x_m}\, A_m$$

  is a finite tree language in the signature

  $$\Sigma \cup \{ f_{\forall \overline{x_i}\, A_i \Rightarrow} \mid 1 \leq i \leq n \} \cup \{ f_{\Rightarrow \exists \overline{x_i}\, A_i} \mid n+1 \leq i \leq m \}$$

  where $f_{\forall \overline{x_i}\, A_i \Rightarrow}$ and $f_{\Rightarrow \exists \overline{x_i}\, A_i}$ resp. has arity $\text{len}(x_i)$.

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  where $f_{\forall \overline{x_i}\, A_i \Rightarrow}$ and $f_{\Rightarrow \exists \overline{x_i}\, A_i}$ resp. has arity $\mathrm{len}(x_i)$.
- **Example.**
  - Let $S_n = P(0), \forall x\, (P(x) \to P(s(x)) \Rightarrow P(s^n(0))$
  - A Herbrand-sequent of S_n is $H_n =$
    $P(0), P(0) \to P(s(0)), \ldots, P(s^{n-1}(0)) \to P(s^n(0)) \Rightarrow P(s^n(0))$
  - H_n is identified with the tree language $L_n =$

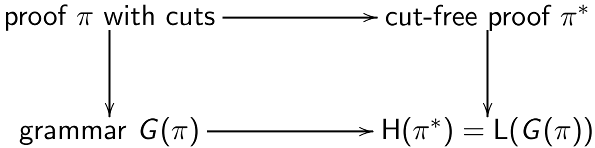
    $$\{ f_{P(0)\Rightarrow},\ f_{\Rightarrow P(s^n(0))},\ f_{A\Rightarrow}(0),\ \ldots,\ f_{A\Rightarrow}(s^{n-1}(0)) \}$$

    where A is $\forall x\, (P(x) \to P(s(x))$.

Proofs and grammars

$$\begin{array}{ccc}
\text{proof } \pi \text{ with cuts} & \longrightarrow & \text{cut-free proof } \pi^* \\
\downarrow & & \downarrow \\
\text{grammar } G(\pi) & \longrightarrow & \mathsf{H}(\pi^*) = \mathsf{L}(G(\pi))
\end{array}$$

Proofs and grammars



Σ_1-cuts $\Longrightarrow$ totally rigid acyclic tree grammars (TRATG)

Regular tree grammars

- **Def.** A *regular tree grammar* is a tuple $G = (N, \Sigma, P, S)$
  - N nonterminal symbols (of arity 0)
  - Σ terminal symbols
  - $S \in N$ starting symbol
  - P production rules $A \to t$ where
    $A \in N$ and $t \in \mathcal{T}(\Sigma \cup N)$

  - $s \Rightarrow_G t$ if $s = r[A]$ and $t = r[u]$ and $A \to u \in P$
  - $\mathrm{L}(G) = \{t \in \mathcal{T}(\Sigma) \mid S \Rightarrow_G^* t\}$ where
    $\Rightarrow_G^*$ reflexive and transitive closure of $\Rightarrow_G$

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  - $s \Rightarrow_G t$ if $s = r[A]$ and $t = r[u]$ and $A \to u \in P$
  - $\mathrm{L}(G) = \{t \in \mathcal{T}(\Sigma) \mid S \Rightarrow_G^* t\}$ where
    $\Rightarrow_G^*$ reflexive and transitive closure of $\Rightarrow_G$
- For $G = (N, \Sigma, P, S)$ define $|G| := |P|$.
- Relation on N: $A <_G^1 B$ if there is $A \to t \in P$ s.t. B occurs in t. Define $<_G$ as transitive closure of $<_G^1$.

TRAT grammars

- **Definition.** A derivation $S \Rightarrow_G^* t$ satisfies *rigidity condition* if it uses at most one A-production for every nonterminal A.
- **Definition.** A totally rigid acyclic tree (TRAT) grammar is an acyclic regular tree grammar $G = (N, \Sigma, P, S)$. Define $\mathrm{L}(G) = \{t \in \mathcal{T}(\Sigma) \mid S \Rightarrow_G^* t \text{ satisfying rigidity condition}\}$.
- *Example.* $S \to f(A, B),\ A \to g(B),\ B \to c \mid d$
  as regular tree grammar:
  $$L = \{f(g(c), c), f(g(c), d), f(g(d), c), f(g(d), d\}$$
  as TRATG:
  $$L = \{f(g(c), c), f(g(d), d)\}$$

Example

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\Rightarrow A(a), A(b)}{\Rightarrow \exists x A(x), A(b)}\;\exists_r}{\Rightarrow \exists x A(x), \exists x A(x)}\;\exists_r}{\Rightarrow \exists x A(x)}\;c_r
\qquad
\dfrac{\dfrac{\dfrac{A(\alpha) \Rightarrow B(f(\alpha))}{A(\alpha) \Rightarrow \exists x B(x)}\;\exists_r
\quad
\dfrac{\dfrac{A(\alpha), B(\beta) \Rightarrow C(g(\alpha, \beta))}{A(\alpha), B(\beta) \Rightarrow \exists x C(x)}\;\exists_r}{A(\alpha), \exists x B(x) \Rightarrow \exists x C(x)}\;\exists_l}{A(\alpha) \Rightarrow \exists x C(x)}\;c_l, cut}{\exists x A(x) \Rightarrow \exists x C(x)}\;\exists_l
}{\Rightarrow \exists x C(x)}\;cut
$$

$G(\pi) = ?$

Example

$$
\dfrac{
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\dfrac{\Rightarrow A(a), A(b)}{\Rightarrow \exists x A(x), A(b)}\,\exists_r
}{\Rightarrow \exists x A(x), \exists x A(x)}\,\exists_r
}{\Rightarrow \exists x A(x)}\,c_r
\qquad
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\quad
\dfrac{
\dfrac{A(\alpha), B(\beta) \Rightarrow C(g(\alpha,\beta))}{A(\alpha), B(\beta) \Rightarrow \exists x C(x)}\,\exists_r
}{A(\alpha), \exists x B(x) \Rightarrow \exists x C(x)}\,\exists_l
}{A(\alpha) \Rightarrow \exists x C(x)}\,c_l, cut
}{\exists x A(x) \Rightarrow \exists x C(x)}\,\exists_l
}{\Rightarrow \exists x C(x)}\,cut
$$

$G(\pi) = \langle \varphi, N, \Sigma, P \rangle$ where $N = \{\varphi, \alpha, \beta\}$ and
$P = \{\varphi \to C(g(\alpha, \beta)),\ \beta \to f(\alpha), \alpha \to a, \alpha \to b\}$

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}{\Rightarrow \exists x C(x)}\,\text{cut}
$$

$\mathrm{G}(\pi) = \langle \varphi, N, \Sigma, P \rangle$ where $N = \{\varphi, \alpha, \beta\}$ and
$P = \{\varphi \rightarrow C(g(\alpha, \beta)),\ \beta \rightarrow f(\alpha), \alpha \rightarrow a, \alpha \rightarrow b\}$

$\mathrm{L}(\mathrm{G}(\pi)) = \{C(g(a, f(a)), C(g(b, f(b)))\}$

Results (in detail)

- For reductions on pruned proofs with at most Σ_2-cuts:
- **Theorem.** If $\pi \to^{\mathsf{ne}} \pi'$ and π' is a normal form then $\mathsf{H}(\pi') = \mathsf{L}(\mathsf{G}(\pi))$.
  - **Lemma.** If $\pi \to^{\mathsf{ne}} \pi'$ then $\mathsf{L}(\mathsf{G}(\pi')) = \mathsf{L}(\mathsf{G}(\pi))$.
- **Theorem.** If $\pi \to \pi'$ and π' is a normal form then $\mathsf{H}(\pi') \subseteq \mathsf{L}(\mathsf{G}(\pi))$.
  - **Lemma.** If $\pi \to \pi'$ then $\mathsf{L}(\mathsf{G}(\pi')) \subseteq \mathsf{L}(\mathsf{G}(\pi))$.

Conclusion

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  - (strong) restriction of reduction
    $\Longrightarrow$ (almost) syntactic confluence ($\mathbf{LK}^{\text{tq}}$)
  - consider (almost) general reduction
    $\Longrightarrow$ weaker notion of confluence (Herbrand-confluence)

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Current / future work based on tree grammars:

- Σ_n-cuts
- Cut-introduction (lemma generation, proof compression)
- Inductive theorem proving
- Proof complexity