
A Linear Logic Framework with Subexponentials

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Linear Logic Basics

Multiplicative Fragment

$$\frac{\Gamma, F, G \longrightarrow H}{\Gamma, F \otimes G \longrightarrow H} \otimes_L$$

$$\frac{\Gamma_1 \longrightarrow F \quad \Gamma_2 \longrightarrow G}{\Gamma_1, \Gamma_2 \longrightarrow F \otimes G} \otimes_R$$

$$\frac{\Gamma_1 \longrightarrow F \quad \Gamma_2, G \longrightarrow H}{\Gamma_1, \Gamma_2, F \multimap G \longrightarrow H} \multimap_L$$

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Contraction and weakening are controlled by the exponentials ! and ?.

$$\frac{\Gamma, !P, !P \longrightarrow G}{\Gamma, !P \longrightarrow G} C$$

$$\frac{\Gamma \longrightarrow G}{\Gamma, ?P \longrightarrow G} W$$

Subexponentials

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Subexponential Signature

$$\langle I, \leq, U \rangle$$

where $U \subseteq I$ and is closed under \leq .

Subexponentials with index $a \in U$ can weaken and contract:

$$\frac{\Gamma, !^a P, !^a P \longrightarrow G}{\Gamma, !^a P \longrightarrow G} \quad C \quad \frac{\Gamma \longrightarrow G}{\Gamma, !^a P \longrightarrow G} \quad W$$

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In fact, given Frank's talk, signatures are of the form:

$$\langle I, \leq, C, W \rangle$$

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Introduction Rules

$$\frac{!^{x_1} F_1, \dots, !^{x_n} F_n \longrightarrow G}{!^{x_1} F_1, \dots, !^{x_n} F_n \longrightarrow !^a G} !^a R$$

$$\frac{!^{x_1} F_1, \dots, !^{x_n} F_n, F \longrightarrow ?^{x_{n+1}} G}{!^{x_1} F_1, \dots, !^{x_n} F_n, ?^a F \longrightarrow ?^{x_{n+1}} G} ?^a L$$

where $a \leq x_i$ for all i .

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$$\frac{!^{x_1} F_1, \dots, !^{x_n} F_n, F \longrightarrow ?^{x_{n+1}} G}{!^{x_1} F_1, \dots, !^{x_n} F_n, ?^a F \longrightarrow ?^{x_{n+1}} G} ?^a L$$

where $a \leq x_i$ for all i .

Theorem: For any subexponential signature, Σ , SELL_Σ admits cut-elimination.

Differences to Linear Logic

- The combination of subexponentials yields an **unbounded number** of **logically distinct prefixes** as one can combine subexponentials with different labels, *e.g.*,
 $!^{l_1}, !^{l_2}, \dots, !^{l_1} ?^{l_1}, !^{l_1} ?^{l_2}, !^{l_1} ?^{l_3}, \dots;$
- Subexponential labels can be **quantified over** leading to new universal and existential quantifiers \multimap and \uplus ;
- The preorder \leq among subexponentials can be constructed using more **involved structures**, *e.g.* c-semirings.

Some Applications

- A framework for **proof systems**;
- A framework for **authorization logics**;
- A framework for **concurrent constraint programming languages**.

Sequents

In **linear logic**, there are two types of formulas **bounded** and **unbounded**. Sequents normally have the form:

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SELL has **as many contexts** as subexponential labels:

$$I = \{l_1, \dots, l_n, \dots, l_{m+n}\} \quad U = \{l_1, \dots, l_n\}$$

$$\underbrace{\Theta_1 \mid \cdots \mid \Theta_n}_{\text{Unbounded}} \mid \underbrace{\Gamma_{n+1} \mid \cdots \mid \Gamma_{n+m} \mid \Gamma}_{\text{Bounded}} \longrightarrow F$$

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LL is an instance of SELL, where $I = U = \{u\}$. For the Linear K system from Frank's talk set $I = \{u\}$ and $U = \emptyset$.

We also have a focused proof system for SELL.

Rules

Bounded contexts are split, while unbounded are contracted:

$$\frac{\Theta_{1..n} \mid \Gamma_{n+1} \mid \cdots \mid \Gamma_{n+m} \mid \Gamma \longrightarrow F_1 \quad \Theta_{1..n} \mid \Gamma'_{n+1} \mid \cdots \mid \Gamma'_{n+m} \mid \Gamma \longrightarrow F_2}{\Theta_{1..n} \mid \Gamma_{n+1}\Gamma'_{n+1} \mid \cdots \mid \Gamma_{n+m}\Gamma'_{n+m} \mid \Gamma\Gamma' \longrightarrow F_1 \otimes F_2}$$

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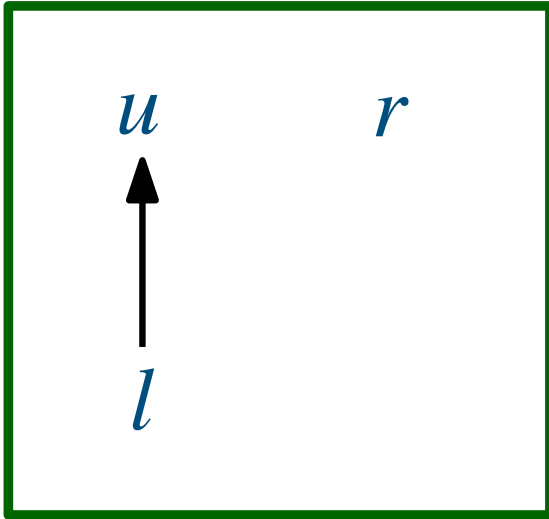
Unbounded contexts may be contracted when necessary:

$$\frac{}{\Theta_{1..n} \mid \cdot \mid \cdots \mid A \mid \cdot \mid \cdot \longrightarrow A} I$$

Preorder

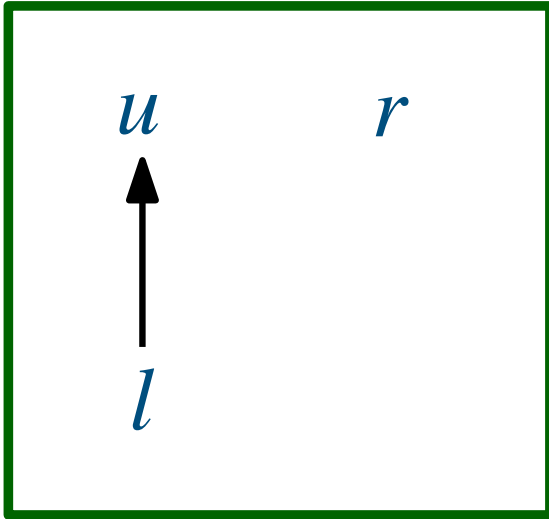
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Consider $I = \{u, l, r\}$, $U = \{u\}$ and the pre-order:



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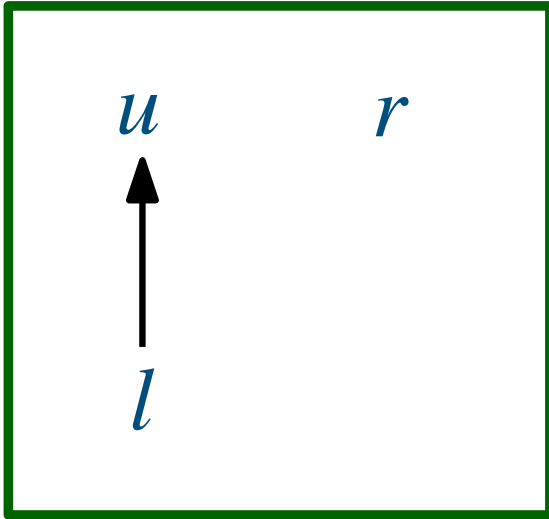
$$\frac{\Theta_u \mid \Gamma_l \mid \cdot \mid \cdot \longrightarrow F}{\Theta_u \mid \Gamma_l \mid \cdot \mid \cdot \longrightarrow !^l F} !^R$$

$$\frac{\cdot \mid \cdot \mid \Gamma_r \mid \cdot \longrightarrow F}{\Theta_u \mid \cdot \mid \Gamma_r \mid \cdot \longrightarrow !^r F} !^R$$

$$\frac{\Theta_u \mid \cdot \mid \cdot \mid \cdot \longrightarrow F}{\Theta_u \mid \cdot \mid \cdot \mid \cdot \longrightarrow !^u F} !^R$$

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$$\frac{\Theta_u \mid \cdot \mid \cdot \mid \cdot \longrightarrow F}{\Theta_u \mid \cdot \mid \cdot \mid \cdot \longrightarrow !^u F} !^R$$

Similarly with left ? introduction rules:

$$\frac{\Theta_u \mid \Gamma_l \mid \cdot \mid G \longrightarrow ?^l F}{\Theta_u \mid ?^l G, \Gamma_l \mid \cdot \mid \cdot \longrightarrow ?^l F} !^R$$

Classical SELL

Sometimes it will be convenient to use the **classical version of SELL**.

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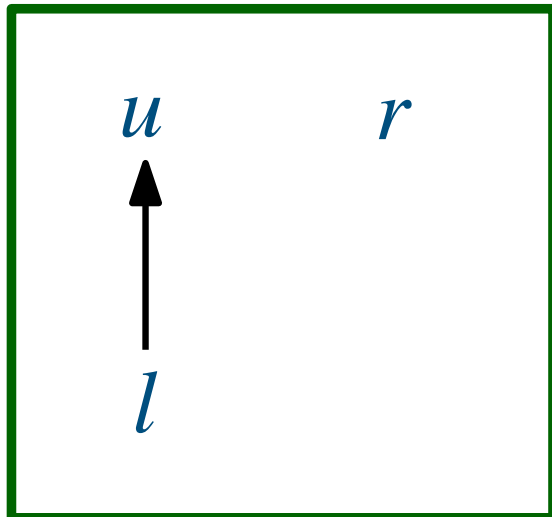
Sequents

$$I = \{l_1, \dots, l_n, \dots, l_{m+n}\}$$

$$U = \{l_1, \dots, l_n\}$$

$$\vdash \Theta_1 \mid \cdots \mid \Theta_n \mid \Gamma_{n+1} \mid \cdots \mid \Gamma_{n+m} \mid \Gamma$$

Rules



$$\frac{}{\vdash \Theta_{1..n} \mid \cdot \mid \cdots \mid A \mid \cdot \mid A^\perp} I$$

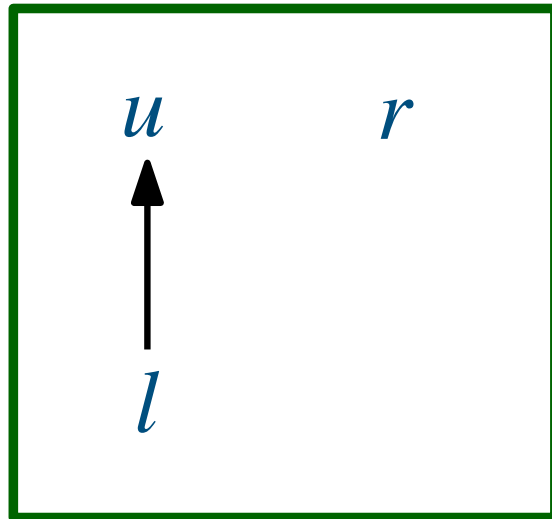
$$\frac{\vdash \Theta_u \mid \Gamma_l \mid \cdot \mid F}{\vdash \Theta_u \mid !^l F, \Gamma_l \mid \cdot \mid \cdot} !_R$$

Agenda

Subexponential Prefixes

- Subexponential Quantification
- Algebras for Subexponential Relations
- Conclusions and Future Work

Prefixes



$$\frac{\Theta_u \mid \Gamma_l \mid \cdot \mid \cdot \longrightarrow F}{\Theta_u \mid \Gamma_l \mid \cdot \mid \cdot \longrightarrow !^l F} !_R$$

$$\frac{\cdot \mid \cdot \mid \Gamma_r \mid \cdot \longrightarrow F}{\Theta_u \mid \cdot \mid \Gamma_r \mid \cdot \longrightarrow !^r F} !_R$$

$$\frac{\Theta_u \mid \cdot \mid \cdot \mid \cdot \longrightarrow F}{\Theta_u \mid \cdot \mid \cdot \mid \cdot \longrightarrow !^u F} !_R$$

- We are able to erase some types of unbounded formulas in the context;
- We are able to check whether only some types of formulas are present in the context.

Prefixes

Classical SELL as a Framework for Proof Systems

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Classical SELL as a Framework for Proof Systems

Object Sequent $F_1, \dots, F_n \longrightarrow G_1, \dots, G_m$

$I = \{u, l, r\}$ $[\cdot], [\cdot] : form \rightarrow o$

Meta Sequent $\vdash \ominus \mid [F_1], \dots, [F_n] \mid [G_1], \dots, [G_m] \mid \cdot$



Encoding of the rules of the proof system, like a logic program.

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Consider the following rule from **the multi-conclusion proof system** for intuitionistic logic:

$$\frac{\Gamma, F \longrightarrow G}{\Gamma \longrightarrow \Delta, F \supset G} \Rightarrow_R$$

Prefixes

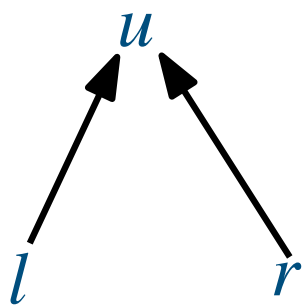
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SELL Encoding

$u, l, r \in U$



$$\exists A. \exists B. [A \supset B]^\perp \otimes !^l (?^l [A] \wp ?^r [B])$$

Prefixes

- We are able to check whether only some types of formulas are present in the context.

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Consider the following rule from **the multi-conclusion proof system** for intuitionistic logic:

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$$\frac{\Gamma, \circ F, F \longrightarrow \circ G}{\Gamma, \circ F \longrightarrow \circ G} \circ L$$

SELL Encoding

$u, l \in U$



Both can store the formula on the r.h.s, but only \circ_r can store a \circ formula.

$$\exists A. [\circ A]^\perp \otimes !^{\circ_r} ?^l [A]$$

Putting this together

Intuitionistic SELL as a Framework for Linear Authorization Logics

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Intuitionistic **SELL** as a Framework for Linear Authorization Logics

Three Families of Modalities [Garg et al.]

$K \text{ says } P$

$K \text{ knows } P$

$K \text{ has } P$

Putting this together

Intuitionistic **SELL** as a Framework for Linear Authorization Logics

Three Families of Modalities [Garg et al.]

K says P

K knows P

K has P

A **lax modality** denoting that the principal K affirms the formula P :

$$\frac{\Gamma, P \longrightarrow K \text{ says } G}{\Gamma, K \text{ says } P \longrightarrow K \text{ says } G} \text{ says}_L$$

$$\frac{\Gamma \longrightarrow P}{\Gamma \longrightarrow K \text{ says } P} \text{ says}_R$$

Putting this together

Intuitionistic **SELL** as a Framework for Linear Authorization Logics

Three Families of Modalities [Garg et al.]

$K \text{ says } P$

$K \text{ knows } P$

$K \text{ has } P$

Since knowledge is unrestricted, one is allowed to contract as well as weaken it:

$$\frac{\Gamma \longrightarrow G}{\Gamma, K \text{ knows } P \longrightarrow G} \quad W$$

$$\frac{\Gamma, K \text{ knows } P, K \text{ knows } P \longrightarrow G}{\Gamma, K \text{ knows } P \longrightarrow G} \quad C$$

Putting this together

Intuitionistic **SELL** as a Framework for Linear Authorization Logics

Three Families of Modalities [Garg et al.]

K says *P*

K knows *P*

K has *P*

An unbounded modality denoting that the principal *K* has the consumable resource *P*:

$$\frac{\Gamma, P \longrightarrow G}{\Gamma, K \text{ has } P \longrightarrow G} \text{ has}_L \quad \frac{\Psi, \Delta \longrightarrow P}{\Psi, \Delta \longrightarrow K \text{ has } P} \text{ has}_R$$

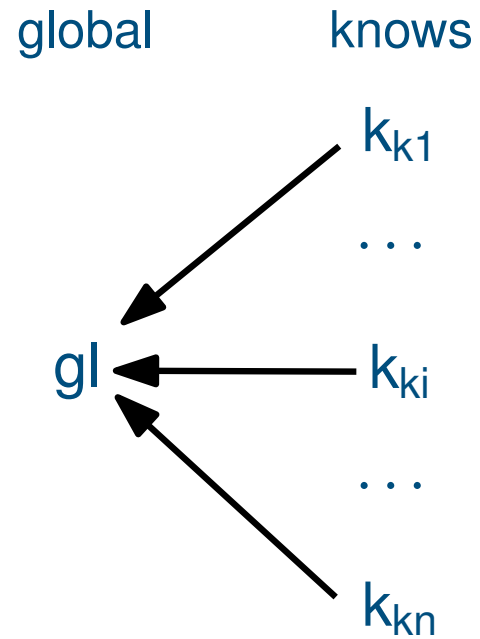
where Ψ contains only formulas of the form *K* knows *F*, while Δ contains only formulas of the form *K* has *F*.

Putting this together

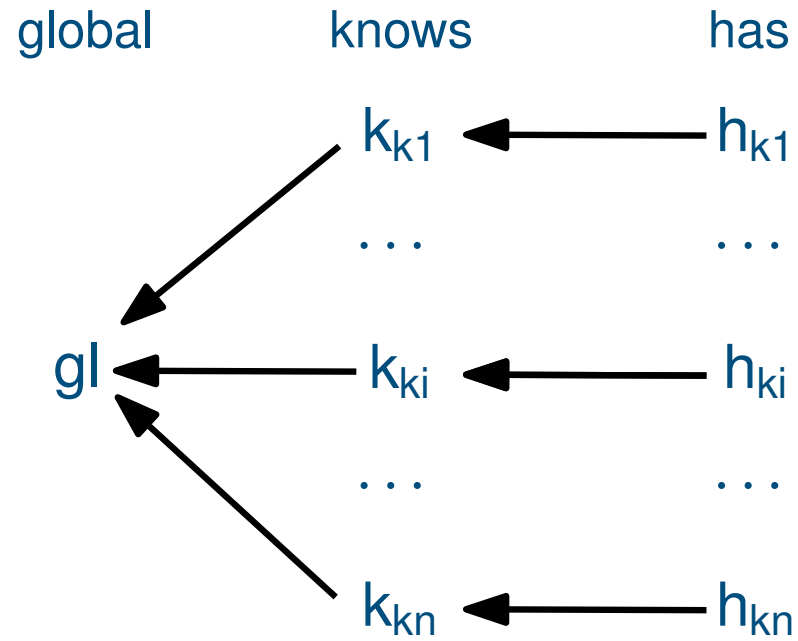
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gl

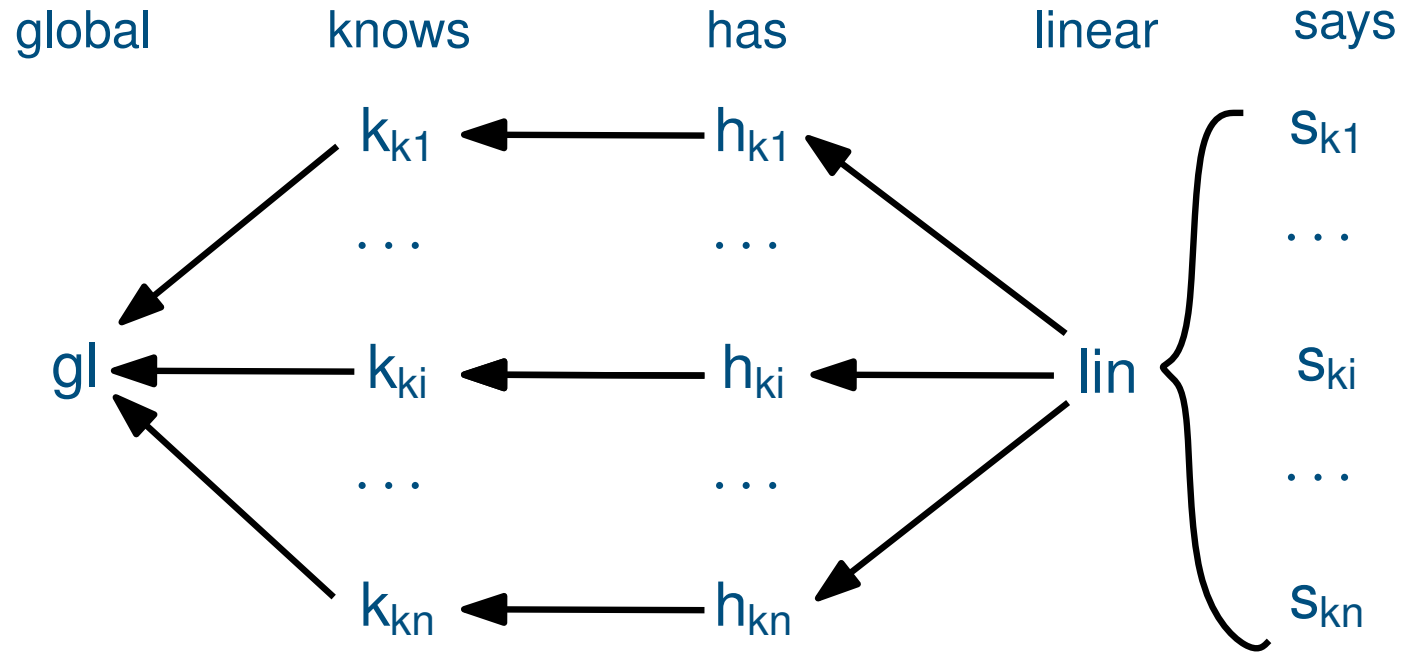
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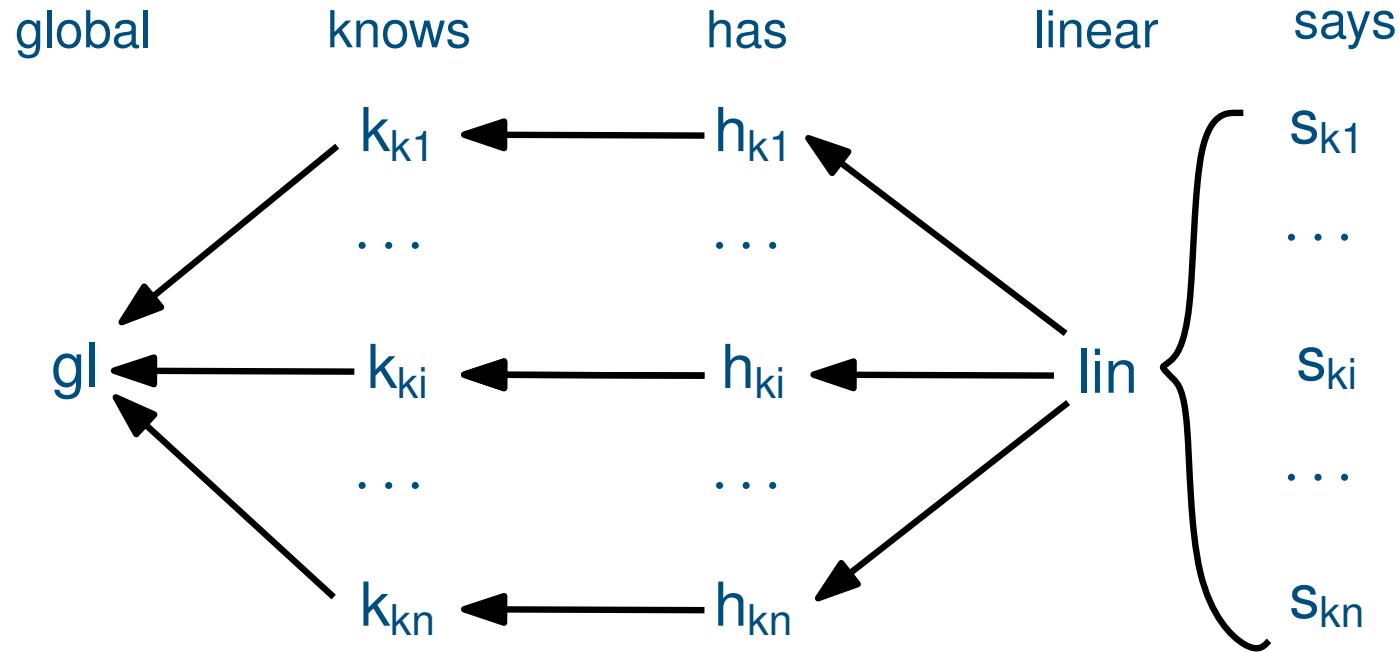
Putting this together



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$$[[F \text{ knows } K]]_L = !^{k_K} [[F]]_L$$

$$[[F \text{ knows } K]]_R = !^{k_K} [[F]]_R$$

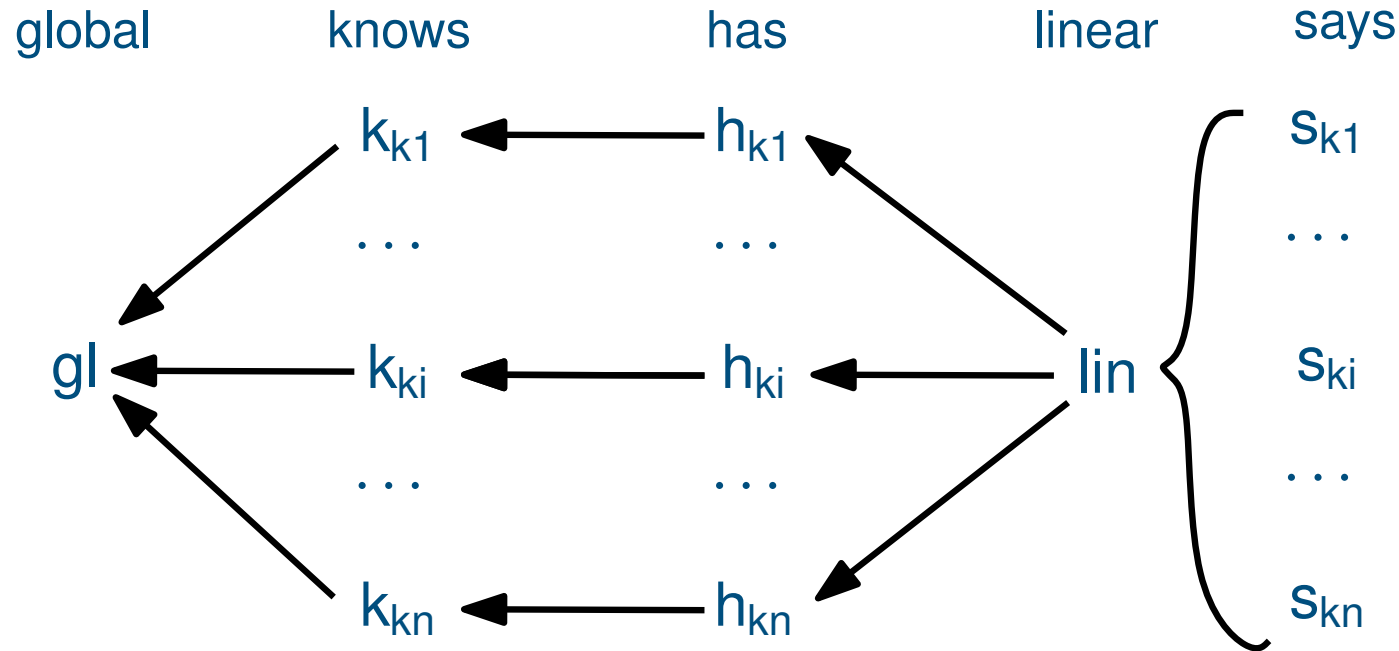
$$[[F \text{ has } K]]_L = !^{h_K} [[F]]_L$$

$$[[F \text{ has } K]]_R = !^{h_K} [[F]]_R$$

$$[[F \text{ says } K]]_L = !^{lin} ?^{s_k} [[F]]_L$$

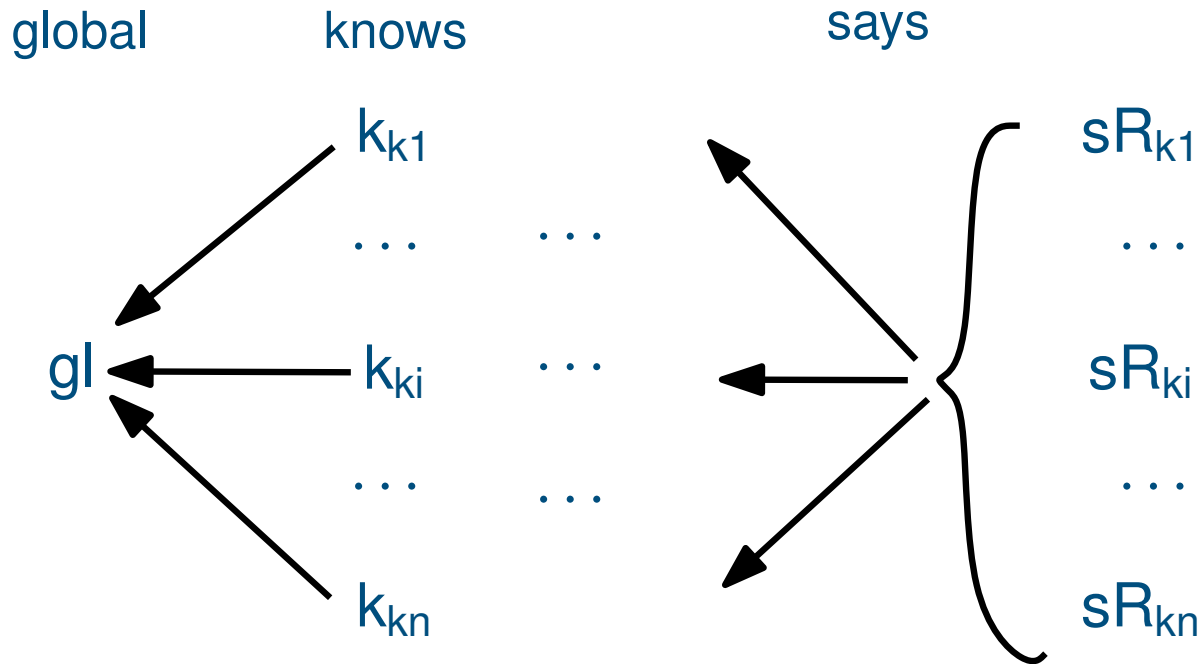
$$[[F \text{ says } K]]_R = ?^{s_k} [[F]]_R$$

Putting this together

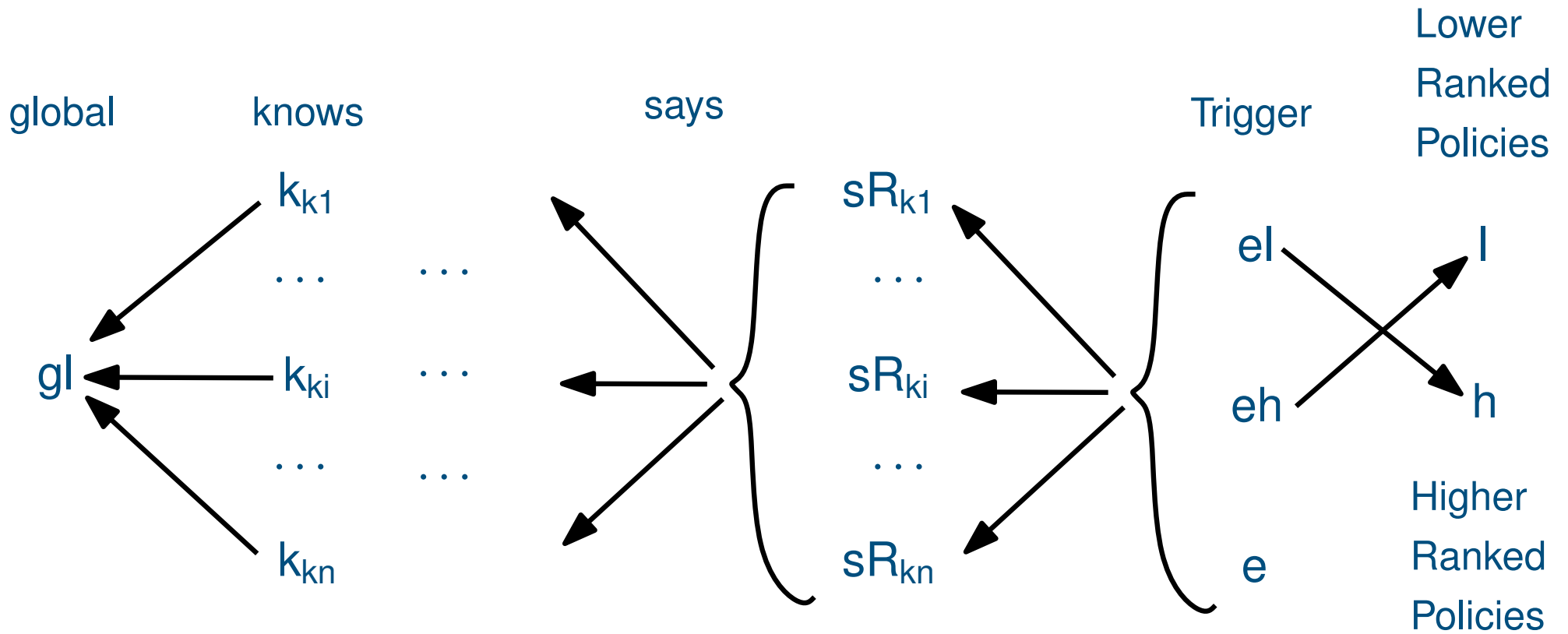


Theorem: The sequent $\Gamma \longrightarrow F$ is provable in linear authorization logic if and only if the sequent $[[\Gamma]]_L \longrightarrow [[F]]_R$ is provable in SELL.

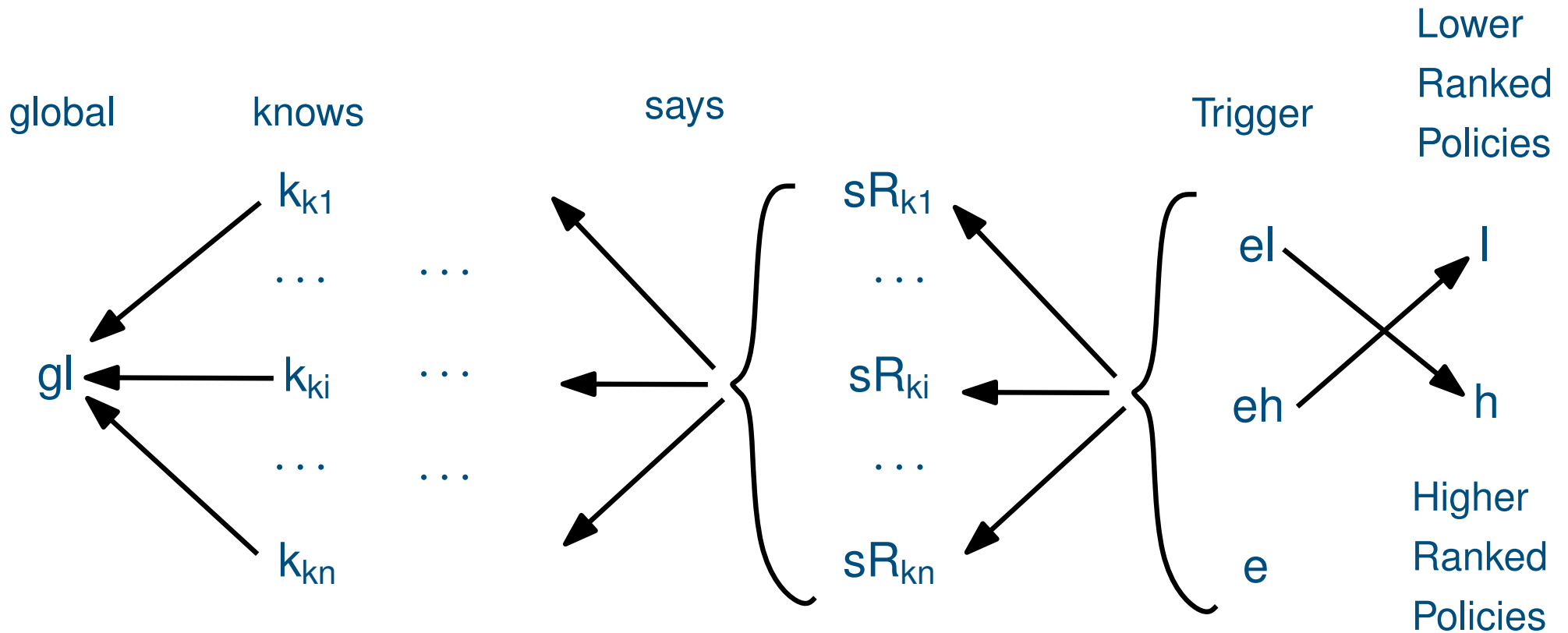
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$$\frac{\frac{\Gamma \longrightarrow F}{\Gamma \longrightarrow !^{el} F} \quad !^{el} R}{\Gamma, !\{\Gamma_L\} \longrightarrow !^{el} F} \quad n \times W$$

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Subexponential Quantification

- Algebras for Subexponential Relations
- Conclusions and Future Work

Adding Subexponential Quantifiers

Subexponential quantification **adds expressiveness to SELL**, but one needs to be **careful that SELL's nice properties**, *e.g.*, cut-elimination and focusing discipline, are still preserved.

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- The idea is to **emulate the cut-elimination reductions** for the first-order quantifiers.
- Quantification may create generic variables, we call **Subexponential Variables**;
- However, subexponentials are organized into a pre-order, so we need more information on the variables. **We add a typing to subexponentials.**

Adding Subexponential Quantifiers

Signatures are of the form:

$$\langle I, \preceq, F, U \rangle$$

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- $F = \{\check{f}_1, \dots, \check{f}_n\}$ is a set of **subexponential index families**. In particular, $\check{f} \in F$ takes an element $a \in I$ and returns a subexponential index $\check{f}(a)$.

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- $U \subseteq \{\check{f}(a) \mid a \in I, \check{f} \in F\}$ is a set of **unbounded subexponentials**. As before, it is upwardly closed with respect to \leq : if $b \leq a$, where $a, b \in I$, and $\check{f}(b) \in U$ then $\check{f}(a) \in U$.

Adding Subexponential Quantifiers

\forall – Universal quantifier;

\exists – Existential quantifier;

Adding Subexponential Quantifiers

\bigcap – Universal quantifier;

\bigcup – Existential quantifier;

$$\frac{\mathcal{A}; \Gamma, P[l/x] \vdash G}{\mathcal{A}; \Gamma, \bigcap x : a.P \vdash G} \bigcap_L$$

$$\frac{\mathcal{A}, l_e : a; \Gamma \vdash P[l_e/x]}{\mathcal{A}; \Gamma \vdash \bigcap x : a.P} \bigcap_R$$

$$\frac{\mathcal{A}, l_e : a; \Gamma, P[l_e/x] \vdash G}{\mathcal{A}; \Gamma, \bigcup x : a.P \vdash G} \bigcup_L$$

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$$\frac{\mathcal{A}; !^{\mathfrak{f}(l_1: a_1)} F_1, \dots, !^{\mathfrak{f}(l_n: a_n)} F_n \longrightarrow G}{\mathcal{A}; !^{\mathfrak{f}(l_1: a_1)} F_1, \dots, !^{\mathfrak{f}(l_n: a_n)} F_n \longrightarrow !^{\mathfrak{f}(l: a)} G}$$

$$\mathfrak{f}(l: a) \leq_{\mathcal{A}} \mathfrak{f}(l_i: a_i)$$

where $\mathfrak{f}(l: a) \leq_{\mathcal{A}} \mathfrak{f}(l_i: a_i)$ means $l_i \in \uparrow l$.

Adding Subexponential Quantifiers

Theorem For any signature Σ , the proof system SELL^{\cap} admits cut-elimination.

SELL^{\cap} also has a complete focused proof system.

Adding Subexponential Quantifiers

Intuitionistic SELL as a Framework for Concurrent
Constraint Programming

Adding Subexponential Quantifiers

Intuitionistic **SELL** as a Framework for Concurrent Constraint Programming

A simple and powerful model of concurrency **tied to logic**:

- Systems are specified by **constraints** representing **partial information** on the variables of the system.
- Agents **tell** and **ask** constraints on a shared **store** of constraints.
- CCP is **parametric** in a Constraint System (e.g. $x > 42 \vdash_{\Delta} x > 0$).

Adding Subexponential Quantifiers

Intuitionistic **SELL** as a Framework for Concurrent Constraint Programming

CCP has been extended to deal with different application domains:

- **tcc**: Reactive and timed systems;
- **lccp**: Linearity and resources;
- **ntcc**: Time, non-determinism and asynchrony;
- **utcc**: Mobility;
- **eccp** and **sccp**: Epistemic and Spatial reasoning.

Adding Subexponential Quantifiers

Intuitionistic SELL as a Framework for Concurrent Constraint Programming

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- **tcc**: Reactive and timed systems;
- **lccp**: Linearity and resources;
- **ntcc**: Time, non-determinism and asynchrony;
- **utcc**: Mobility;
- **eccp** and **sccp**: Epistemic and Spatial reasoning.

All these systems can be encoded in SELL^h. In fact, we show how to combine some of them.

Adding Subexponential Quantifiers

Intuitionistic **SELL** as a Framework for Concurrent Constraint Programming

- $!^s P$ is **located** at s (epistemic and temporal);
- $!^s ?^s P$ is **confined** to s (spatial);
- $\mathcal{M}l : a P - P$ can **move** to locations below (outside) a (mobility).

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All our encodings have a **strong level of adequacy**: proof search and the execution of encoded programs **match exactly**.

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More details in our CONCUR 2013 paper and in the submitted journal version.

Agenda

- Subexponential Prefixes
- Subexponential Quantification

Algebras for Subexponential Relations

- Conclusions and Future Work

Algebra for Subexponential Relations

Until now, \leq was **quite simple**. We can add more structure it **to capture even more computational behaviors**.

Algebra for Subexponential Relations

C-Semiring is a tuple $\langle \mathcal{A}, +, \times, \perp_A, \top_A \rangle$

- $+$: commutative, associative, **idempotent**, \perp_A -unit, \top_A -**absorbing**
- \times is associative, **commutative**, distribute over $+$, \top_A -unit, \perp_A -absorbing

Let \leq_A be defined as $a \leq_A b$ iff $a + b = b$. Then, $\langle \mathcal{A}, \leq_A \rangle$ is a complete lattice where:

- $+$ and \times are monotone on \leq_A , $+$ is the *lub operator*.

If \times is idempotent, then

- $\langle \mathcal{A}, \leq_A \rangle$ is a complete distribute lattice, \times is its *glb*.

Algebra for Subexponential Relations

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Choses the "best" valuation.

Combines constraints

Algebra for Subexponential Relations

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Combines constraints

- Crisp: $S_c = \langle \{\text{true}, \text{false}\}, \vee, \wedge, \text{false}, \text{true} \rangle$
- Fuzzy: $S_F = \langle [0, 1], \max, \min, 0, 1 \rangle$ – Preferences
- Probabilistic: $S_P = \langle [0, 1], \max, \times, 0, 1 \rangle$
- Weighted: $S_w = \langle \mathcal{R}^-, \max, +, -\infty, 0 \rangle$ – Costs

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An example of Fuzzy constraints:

x	y	$x < y$	$x > 1$	$c_1 \otimes c_2$
1	1	0.5	0.2	0.2
1	2	1.0	0.2	0.2
2	1	0.2	1.0	0.2
2	2	0.5	1.0	0.5

$\sum v_i = 0.5$. Best solution=0.5

Algebra for Subexponential Relations

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More details in our ICLP 2014 paper. In our TCS paper,
we show how soft constraints can be combined with
spatial, epistemic and temporal modalities.

Agenda

- Subexponential Prefixes
- Subexponential Quantification
- Algebras for Subexponential Relations

Conclusions and Future Work

Conclusions and Future Work

- We reviewed SELL a linear logic framework with subexponentials and its extensions.
- We briefly explained how SELL can be used as a framework for Proof Systems, Authorization Logics, and CCP.

Conclusions and Future Work

As future work, we are investigating:

- **Verification of SELL specifications:** Linear logic does help in proving properties about proof systems, such as cut-elimination, when rules permute, etc. More is needed to understand how one can profit when specifying other types of systems.
- **Other algebras for \leq :** Investigate mechanisms to combine modalities in a more systematic fashion.
- **Other forms of quantification:** There seems to be a number of forms of quantifying subexponentials. We need to understand these better.
- **Other applications:** Cyber-Physical security protocols, verification of drone strategies.

Questions