λ-Calculi and Confluence

from A to Z

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Outline

- Confluence of λ-calculus was hard
  - Church & Rosser's seminal paper in 1936
- Confluence of λ-calculus is no longer hard
  - Tait & Martin-Löf, Shanker, Pfenning, McKinna & Pollack, and Takahashi, …
  - Dehornoy and van Oostrom's Z theorem
- Confluence of λ-calculi with permutations is still hard
  - Compositional Z makes it easy

This talk

- Brief history of confluence of λ-calculus
- Brief summary of Z theorem
- Compositional Z, a new proof technique,
    - simpler proof of λ with permutation rules

History of Confluence of λ

λ_β

- Terms

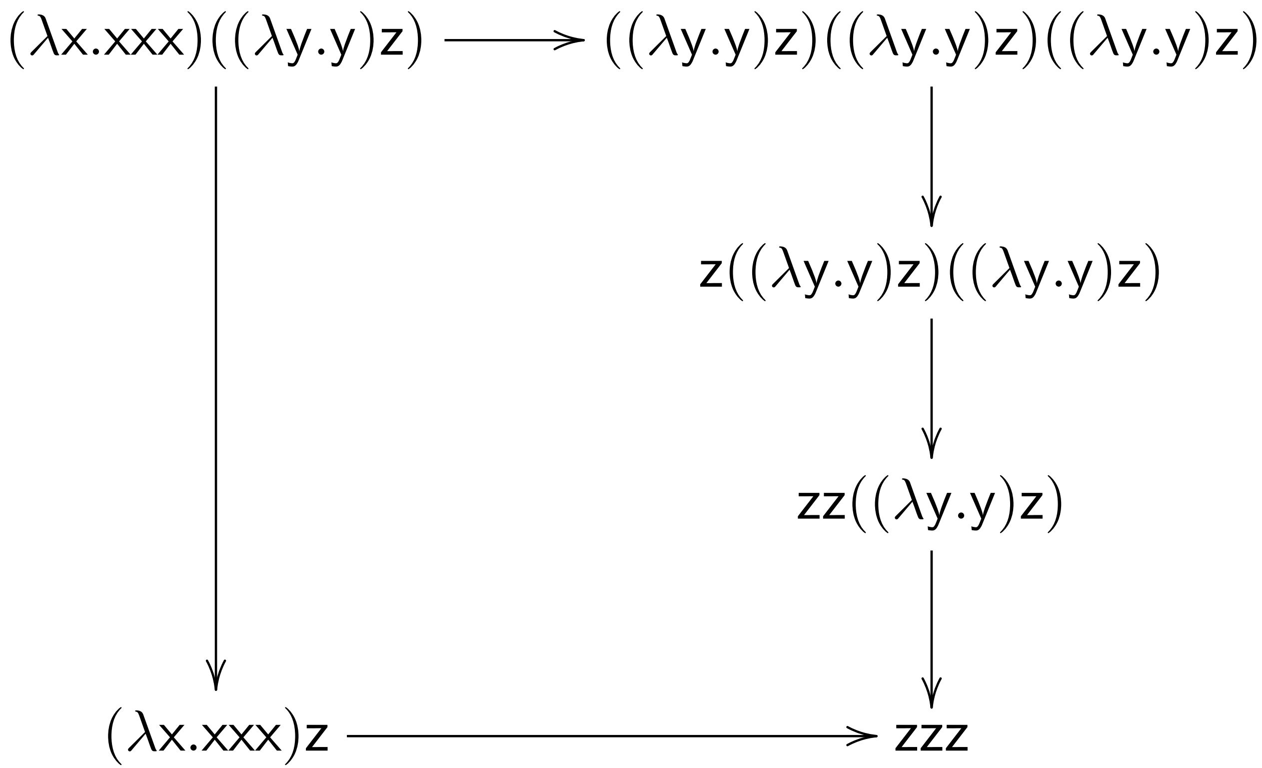
$$M, N ::= x \mid \lambda x.M \mid MN$$

- Reduction rules

$$(\lambda x.M)N \quad \to_\beta \quad M[x := N]$$

Confluence of λ_β is hard

- β is not SN (Newman's lemma cannot be applied)
- (one-step) β does not satisfy diamond property



History of Confluence of λ_β

- Church and Rosser (1936) "*Some Properties of Conversion*"
  - residuals of redexes
- Tait and Martin-Löf (19??)
  - parallel reduction
- Takahashi (1995) "*Parallel Reduction in λ-Calculus*"
  - maximum parallel reduction
- Dehornoy and van Oostrom (2008)
  - Z theorem

Churc (1936)

SOME PROPERTIES OF CONVERSION*

BY

ALONZO CHURCH AND J. B. ROSSER

Our purpose is to establish the properties of conversion which are expressed in Theorems 1 and 2 below. We shall consider first conversion defined by Church's Rules I, II, III† and shall then extend our results to several other kinds of conversion.‡

1. **Conversion defined by Church's Rules I, II, III.** In our study of conversion we are particularly interested in the effects of Rules II and III and consider that applications of Rule I, though often necessary to prevent confusion of free and bound variables, do not essentially change the structure of a formula. Hence we shall omit mention of applications of Rule I whenever it seems that no essential ambiguity will result. Thus when we speak of replacing $\{\lambda x. M\}(N)$§ by $S_N^x M|$ it shall be understood that any applications of I

THEOREM 1. *If **A** conv **B**, there is a conversion from **A** to **B** in which no expansion precedes any reduction.*

B," shall mean that it is possible to go from ***A*** to ***B*** by a single reduction. "***A*** red ***B***," read "***A*** is reducible to ***B***," shall mean that it is possible to go from ***A*** to ***B*** by one or more reductions.|| "***A*** conv-I ***B***," read "***A*** conv ***B*** by

$$\mathsf{A} = \mathsf{B} \quad \Longrightarrow \quad \mathsf{A} \rightarrow \cdot \rightarrow \cdots \rightarrow \cdot \leftarrow \cdots \leftarrow \cdot \leftarrow \mathsf{B}$$

for the foundation of logic, Annals of Mathematics, (2), vol. 33 (1932), pp. 346–366 (see pp. 355–356), as modified by S. C. Kleene, *Proof by cases in formal logic*, Annals of Mathematics, (2), vol. 35 (1934), pp. 529–544 (see p. 530). We assume familiarity with the material on pp. 349–355 of Church's paper and in §§1, 2, 3, 5 of Kleene's paper. We shall refer to the latter paper as "Kleene."

‡ The authors are indebted to Dr. S. C. Kleene for assistance in the preparation of this paper, in particular for the detection of an error in the first draft of it and for the suggestion of an improvement in the proof of Theorem 2.

§ Note carefully the convention at the beginning of §3, Kleene, which we shall constantly use.

|| Our use of "conv" allows us to write "***A*** conv ***B***" even in the case that no applications of I, II, or III are made in going from ***A*** to ***B*** and ***A*** is the same as ***B***. But we write "***A*** red ***B***" only if there is at least one reduction in the process of going from ***A*** to ***B*** by applications of I and II, and use the notation "***A*** conv-I-II ***B***" if we wish to allow the possibility of no reductions.

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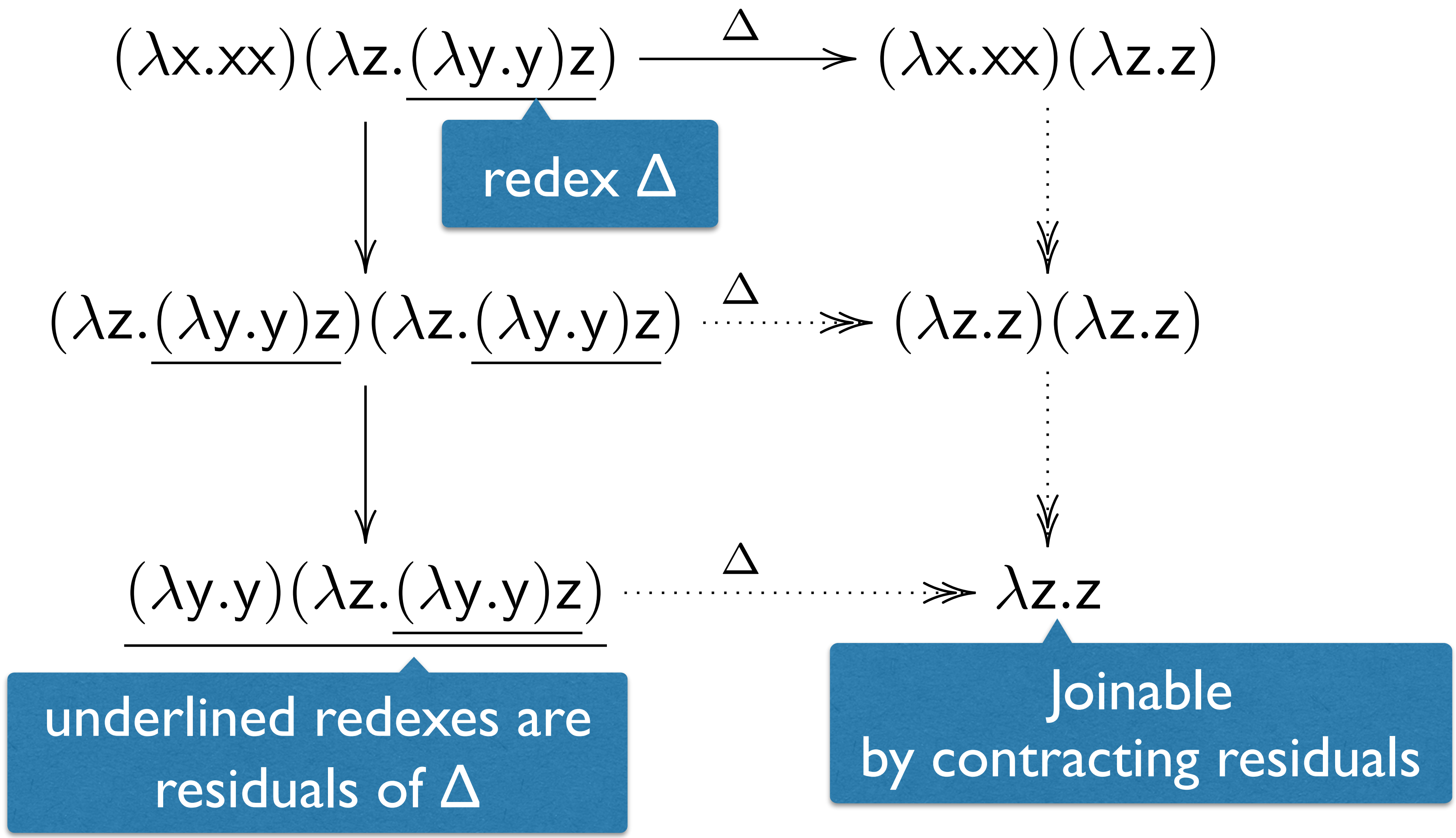
Church and Rosser (1936)

residuals of redexes



Tait and Martin-Löf

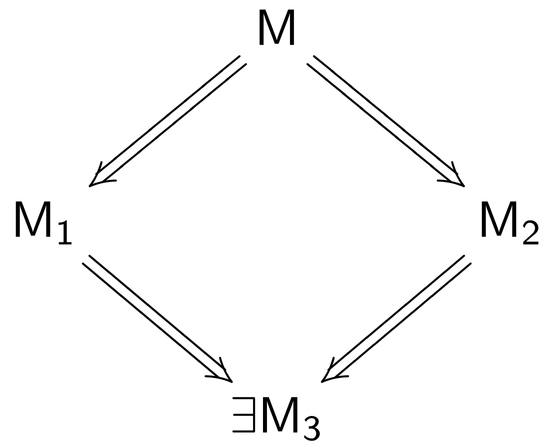
parallel reduction

$$\frac{}{x \Rrightarrow x} \qquad \frac{M \Rrightarrow M' \quad N \Rrightarrow N'}{MN \Rrightarrow M'N'}$$

$$\frac{M \Rrightarrow M'}{\lambda x.M \Rrightarrow \lambda x.M'} \qquad \frac{M \Rrightarrow M' \quad N \Rrightarrow N'}{(\lambda x.M)N \Rrightarrow M'[x := N']}$$

$$\rightarrow_\beta \; \subseteq \; \Rrightarrow \; \subseteq \; \rightarrow_\beta^*$$

and

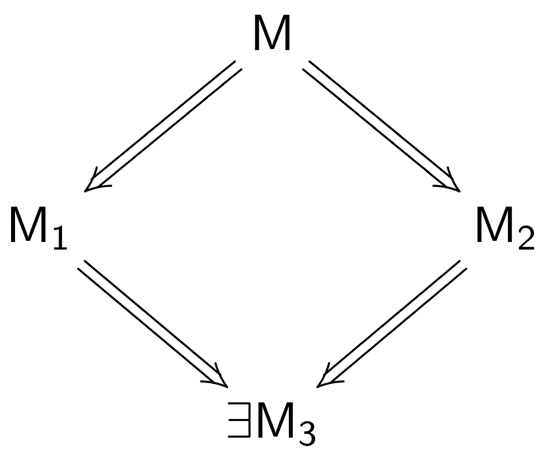


diamond property

Takahashi (1995)

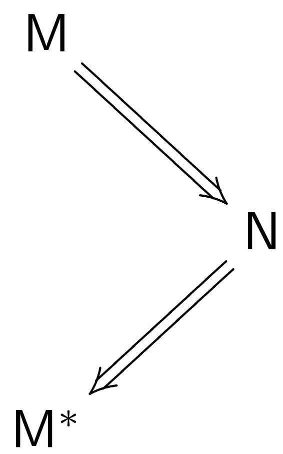
maximum parallel reduction

$$
\begin{aligned}
x^* &= x \\
(\lambda x.M)^* &= \lambda x.M^* \\
((\lambda x.M)N)^* &= M^*[x := N^*] \\
(MN)^* &= M^* N^* \qquad \text{(M is not abst.)}
\end{aligned}
$$



diamond property

follows from

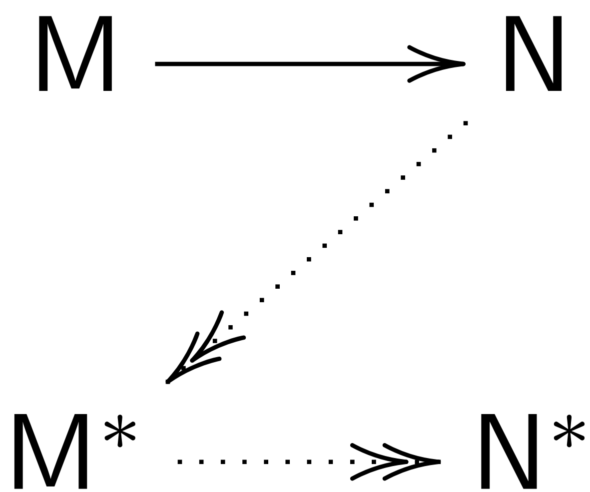


angle property

Dehornoy and van Oostrom (2008)

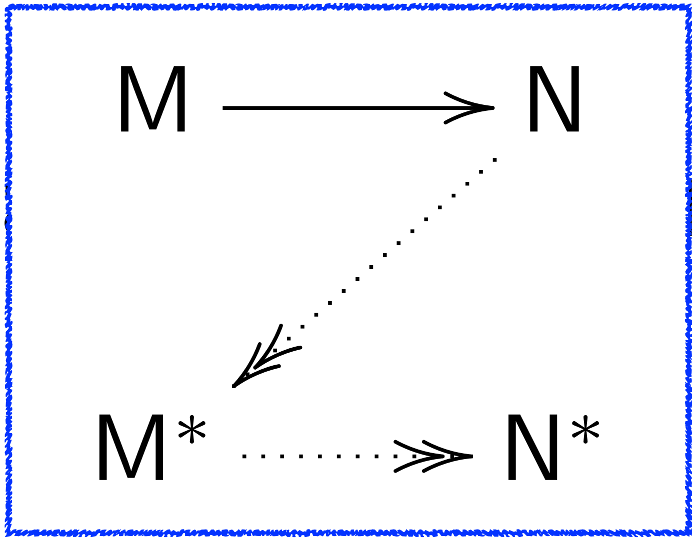
Z theorem

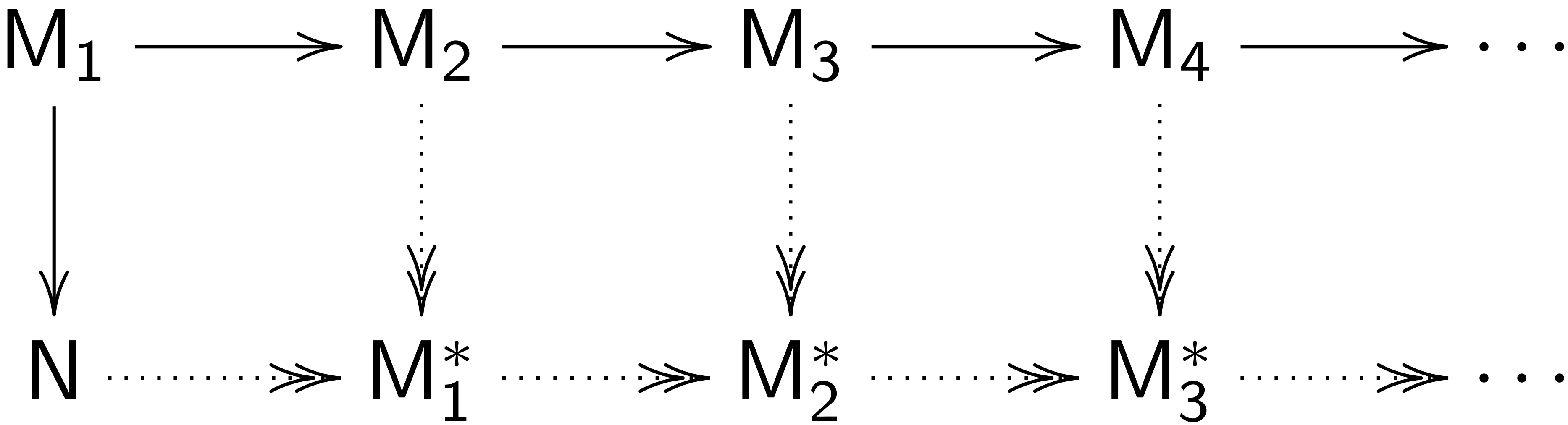
If we find a mapping $(\cdot)^*$ s.t.



then the reduction system is confluent

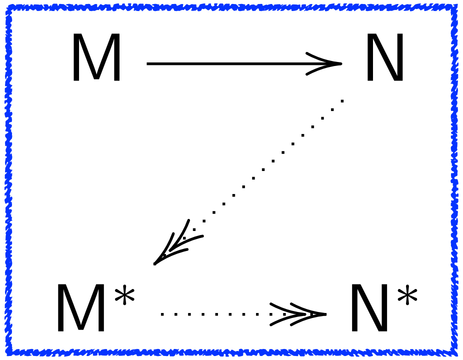
Dehornoy … rom (2008)





A Brief Summary of Z

How nice is Z?



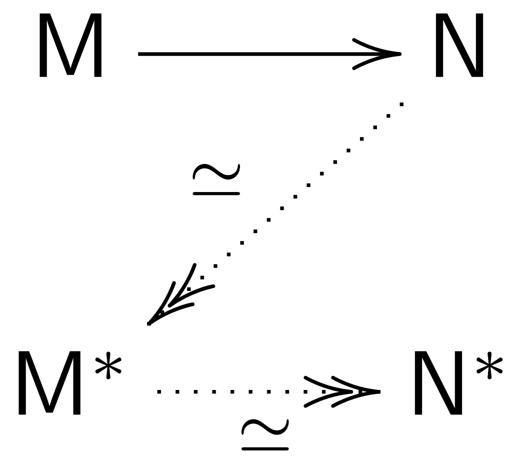
- Z can be proved by induction on one-step reduction
- Z does not require parallel reductions
- Z is easy to apply to reduction systems on quotient sets
  - Z property modulo [Accattoli&Kesner 2012]
    Confluence of λ divided by an equality of associativity
- Z enables us to extend confluence proofs
  - Compositional Z [N&Fujita 2015]
    (in the latter part of this talk)

Z property modulo

- $\rightarrow$: reduction on A induces a reduction on A/$\simeq$

$$M \rightarrow_{\simeq} N \quad \text{iff} \quad M \simeq \cdot \rightarrow \cdot \simeq N$$

- [Accattoli&Kesner2012]
  $\rightarrow_{\simeq}$ is confluent if
  there exists $(\cdot)^*$ well-defined on A/$\simeq$ such that



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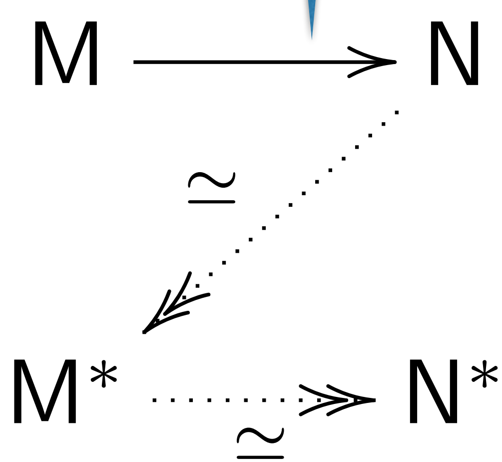
weaker variant of Z



Z property modulo

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  $\rightarrow_{\simeq}$ is confluent if
  there exists $(\cdot)^*$ well-defined on A/$\simeq$ such that

weaker variant of Z

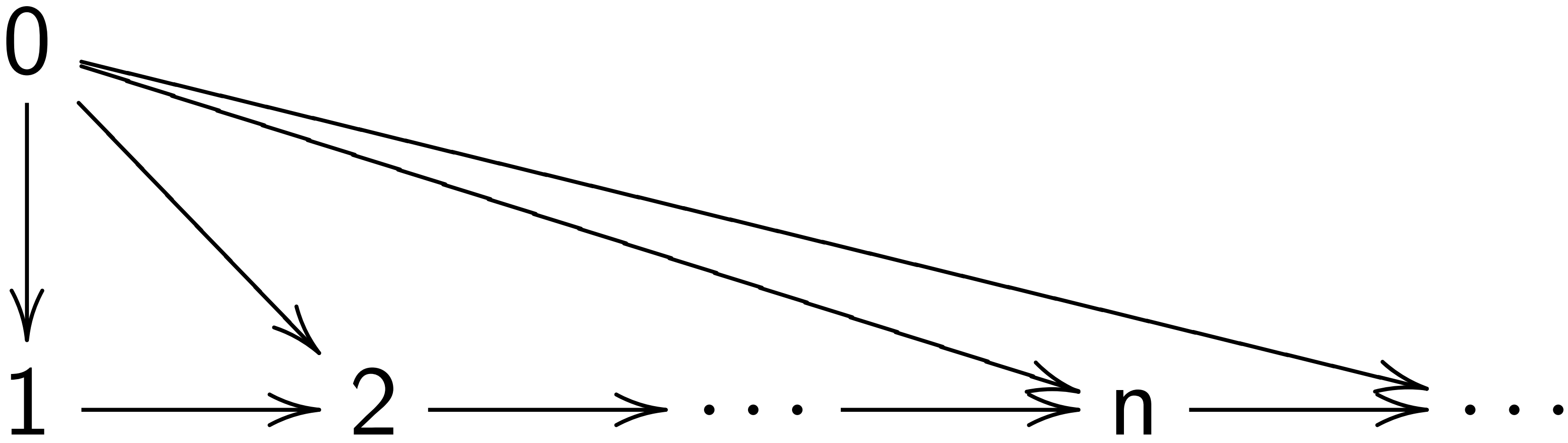
$$\begin{array}{ccccccc} M & \simeq & M' & \longrightarrow & N' & \simeq & N \\ & & & \swarrow_{\simeq} & & & \\ M^* & = & M'^* & \twoheadrightarrow_{\simeq} & N'^* & = & N^* \end{array}$$

Is Z universal?

- Question: For any confluent system, is there a mapping satisfying Z?

- Two extreme (and trivial) answers:
  - YES, if weakly normalizable
    - take M^* as the normal form of M
  - NO, in general
    - there is some counterexamples

Confluent, but not Z



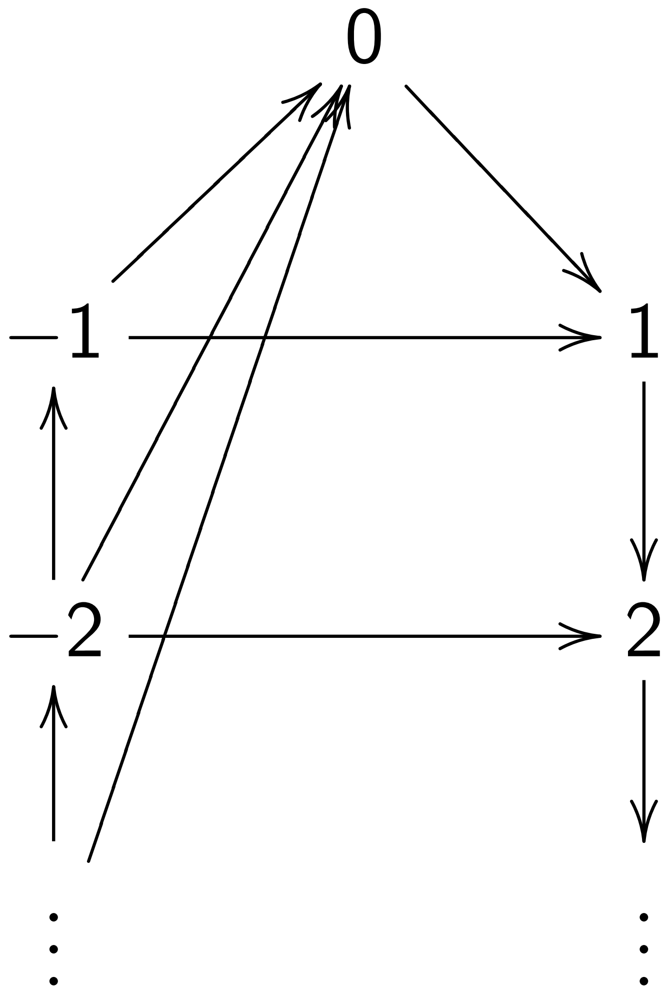
If $0^*=n$,

since $0 \rightarrow n+1$,

$n+1 \rightarrow^* n$ is required by the Z property

Therefore, there is no mapping satisfying Z

Confluent, finitely branching, but not Z



If 0^{*}=n ($\geqq$1),
since -(n+1) → 0,
$(-(n+1))^{*} \to^{*}$ n is required
but $(-(n+1))^{*} \geqq$ n+1 since -(n+1) → n+1

Therefore, there is no mapping satisfying Z

λ-Calculi and Z

Confluence of λ by Z

- Dehornoy and van Oostrom (2008) λ_β, $\lambda_{\beta\eta}$
- Accattoli & Kesner (2012) λ divided by associativity
  - Z property modulo
- Komori, Matsuda, & Yamakawa (2013) λ_β, $\lambda_{\beta\eta}$
- N & Nagai (2014) Λμ
- N & Fujita (2015) $\lambda_{\beta\pi}$, $\lambda\mu_{\beta\pi}$ — π = permutative conversion
  - Compositional Z

Confluence of λ_β by Z

- The maximum parallel reduction is Z

$$
\begin{aligned}
x^* &= x \\
(\lambda x.M)^* &= \lambda x.M^* \\
((\lambda x.M)N)^* &= M^*[x := N^*] \\
(MN)^* &= M^*N^* \qquad \text{(M is not abst.)}
\end{aligned}
$$

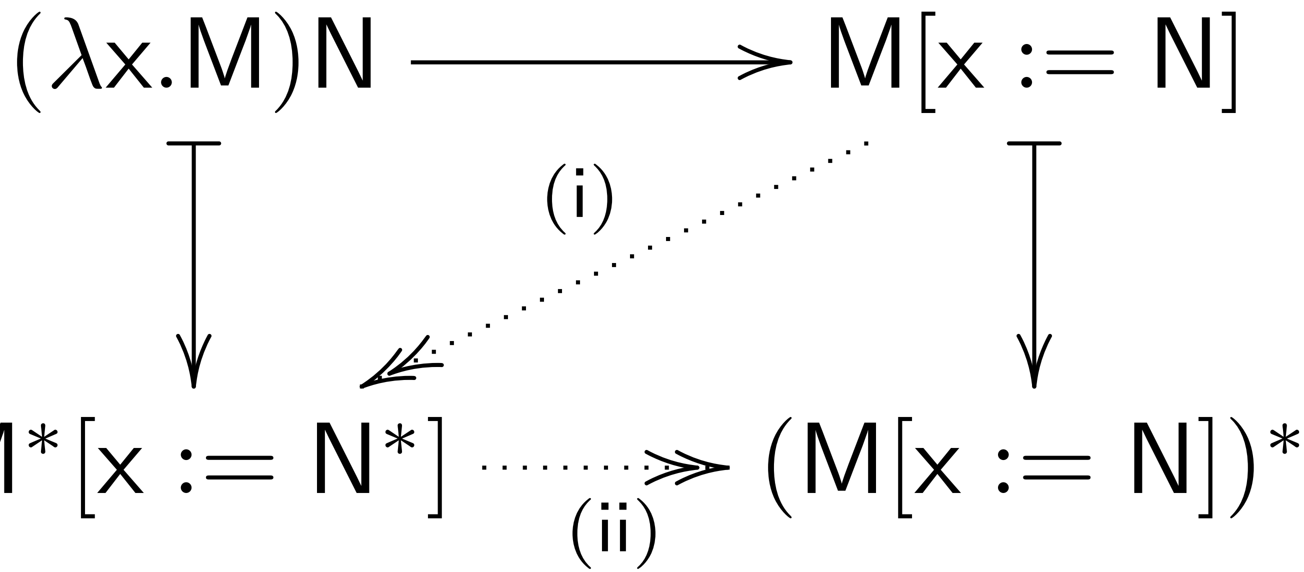
- Key lemmas

$$M \to^* M^* \qquad \text{(i)}$$

$$M^*[x := N^*] \to^* (M[x := N])^* \qquad \text{(ii)}$$

Confluence of λ_β by Z

- Proof of the base case



- Key lemmas

$$M \to^* M^* \qquad \textbf{(i)}$$

$$M^*[x := N^*] \to^* (M[x := N])^* \qquad \textbf{(ii)}$$

Permutative conversion

- for natural deduction with $\vee$ and $\exists$ [Prawitz 1965]
- exchanges order of elimination rules
  - for normal proofs to have good properties such as the subformula property
- makes confluence proofs much harder [Ando 2003]

Exchanging E-Rules

$$
\dfrac{
  \dfrac{
    \overset{\vdots\, P}{\Gamma \vdash A_1 \vee A_2}
    \quad
    \overset{\vdots\, Q_1}{\Gamma, A_1 \vdash B \to C}
    \quad
    \overset{\vdots\, Q_2}{\Gamma, A_2 \vdash B \to C}
  }{\Gamma \vdash B \to C}\,(E_{\vee})
  \qquad
  \overset{\vdots\, R}{\Gamma \vdash B}
}{\Gamma \vdash C}\,(E_{\to})
$$

$$\Big\Downarrow\ \pi$$

$$
\dfrac{
  \overset{\vdots\, P}{\Gamma \vdash A_1 \vee A_2}
  \quad
  \dfrac{
    \overset{\vdots\, Q_1}{\Gamma, A_1 \vdash B \to C}
    \quad
    \overset{\vdots\, R}{\Gamma \vdash B}
  }{\Gamma, A_1 \vdash C}\,(E_{\to})
  \quad
  \dfrac{
    \overset{\vdots\, Q_2}{\Gamma, A_2 \vdash B \to C}
    \quad
    \overset{\vdots\, R}{\Gamma \vdash B}
  }{\Gamma, A_1 \vdash C}\,(E_{\to})
}{\Gamma \vdash C}\,(E_{\vee})
$$

Exchangi

(case P with $x_1 \rightarrow Q_1$ | $x_2 \rightarrow Q_2$)R
=
$P[x_1.Q_1, x_2.Q_2]R$

$$\dfrac{\dfrac{\dfrac{\vdots\ P}{\Gamma \vdash A_1 \vee A_2} \quad \dfrac{\vdots\ Q_1}{\Gamma, A_1 \vdash B \to C} \quad \dfrac{\vdots\ Q_2}{\Gamma, A_2 \vdash B \to C}}{\Gamma \vdash B \to C}\,(E_\vee) \qquad \dfrac{\vdots\ R}{\Gamma \vdash B}}{\Gamma \vdash C}\,(E_\to)$$

π

π

$$\dfrac{\dfrac{\vdots\ P}{\Gamma \vdash A_1 \vee A_2} \quad \dfrac{\dfrac{\vdots\ Q_1}{\Gamma, A_1 \vdash B \to C} \quad \dfrac{\vdots\ R}{\Gamma \vdash B}}{\Gamma, A_1 \vdash C}\,(E_\to) \quad \dfrac{\dfrac{\vdots\ Q_2}{\Gamma, A_2 \vdash B \to C} \quad \dfrac{\vdots\ R}{\Gamma \vdash B}}{\Gamma, A_1 \vdash C}\,(E_\to)}{\Gamma \vdash C}\,(E_\vee)$$

$P[x_1.Q_1R, x_2.Q_2R]$

λβπ

- Terms and eliminators

$$M, N ::= x \mid \lambda x.M \mid \iota_1 M \mid \iota_2 M \mid Me$$

$$e ::= M \mid [x_1.N_1, x_2.N_2]$$

- Reduction rules

$$\begin{aligned}
(\lambda x.M)N &\rightarrow_\beta M[x := N] \\
(\iota_i M)[x_1.N_1, x_2.N_2] &\rightarrow_\beta N_i[x_i := M] \\
M[x_1.N_1, x_2.N_2]e &\rightarrow_\pi M[x_1.N_1 e, x_2.N_2 e]
\end{aligned}$$

λβπ, for simplicity

- Terms and eliminators

$$M, N ::= x \mid \lambda x.M \mid \iota M \mid Me$$

$$e ::= M \mid [x.N]$$

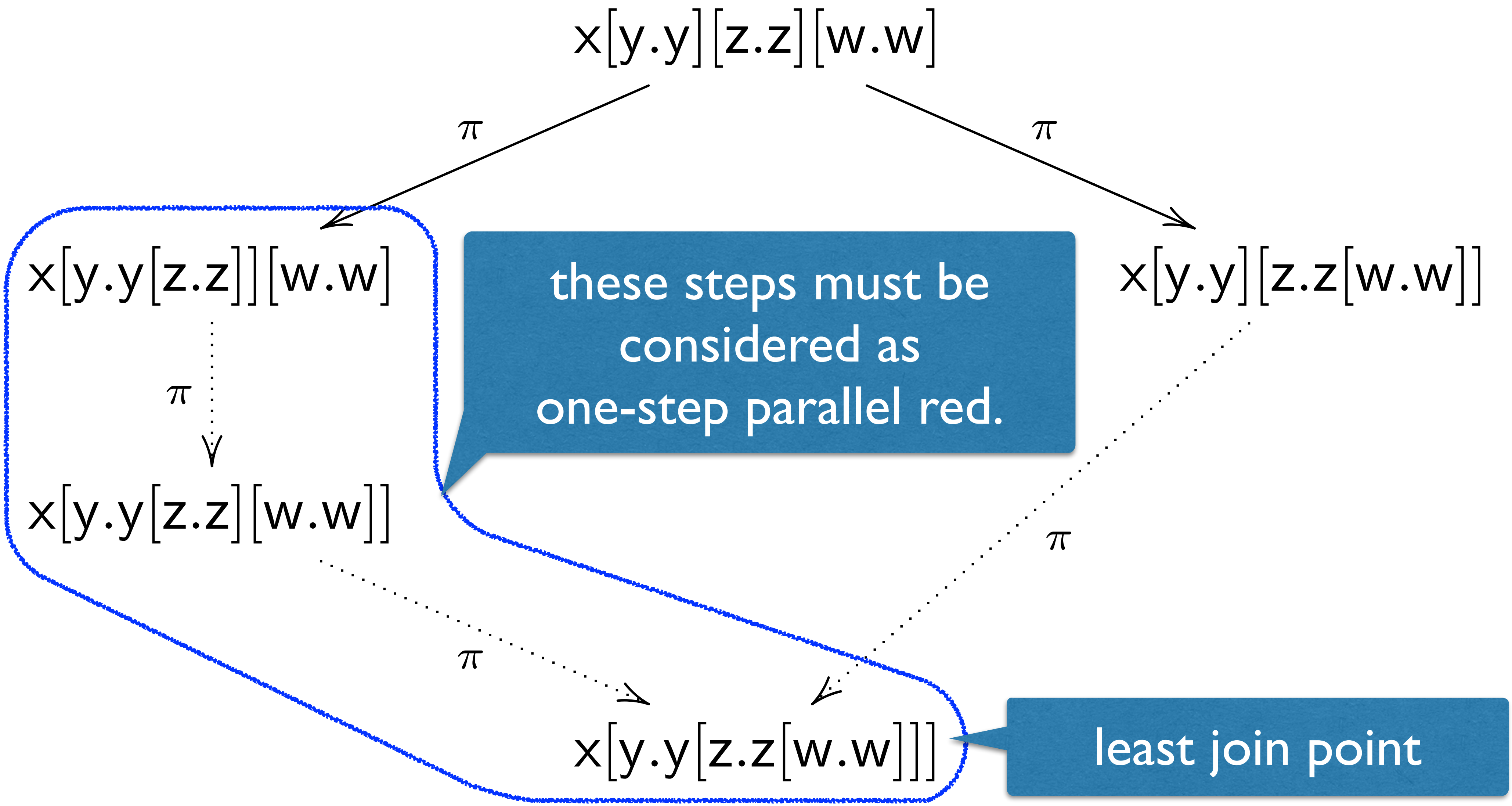
- Reduction rules

$$\begin{array}{rcl} (\lambda x.M)N & \to_\beta & M[x := N] \\ (\iota M)[x.N] & \to_\beta & N[x := M] \\ M[x.N]e & \to_\pi & M[x.Ne] \end{array}$$

Where are difficulties?

- Parallel reduction for π-reduction
- Maximum complete development for the combination of β- and π-reductions

Parallel reduction for π?



Generalizing parallel reduction

- As one-step parallel reduction,
  we have to admit all of the following

$$x[y.y][z.z][w.w] \Rightarrow x[y.y[z.z]][w.w]$$

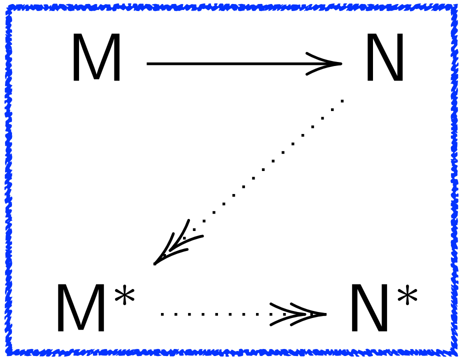
$$x[y.y][z.z][w.w] \Rightarrow x[y.y][z.z[w.w]]$$

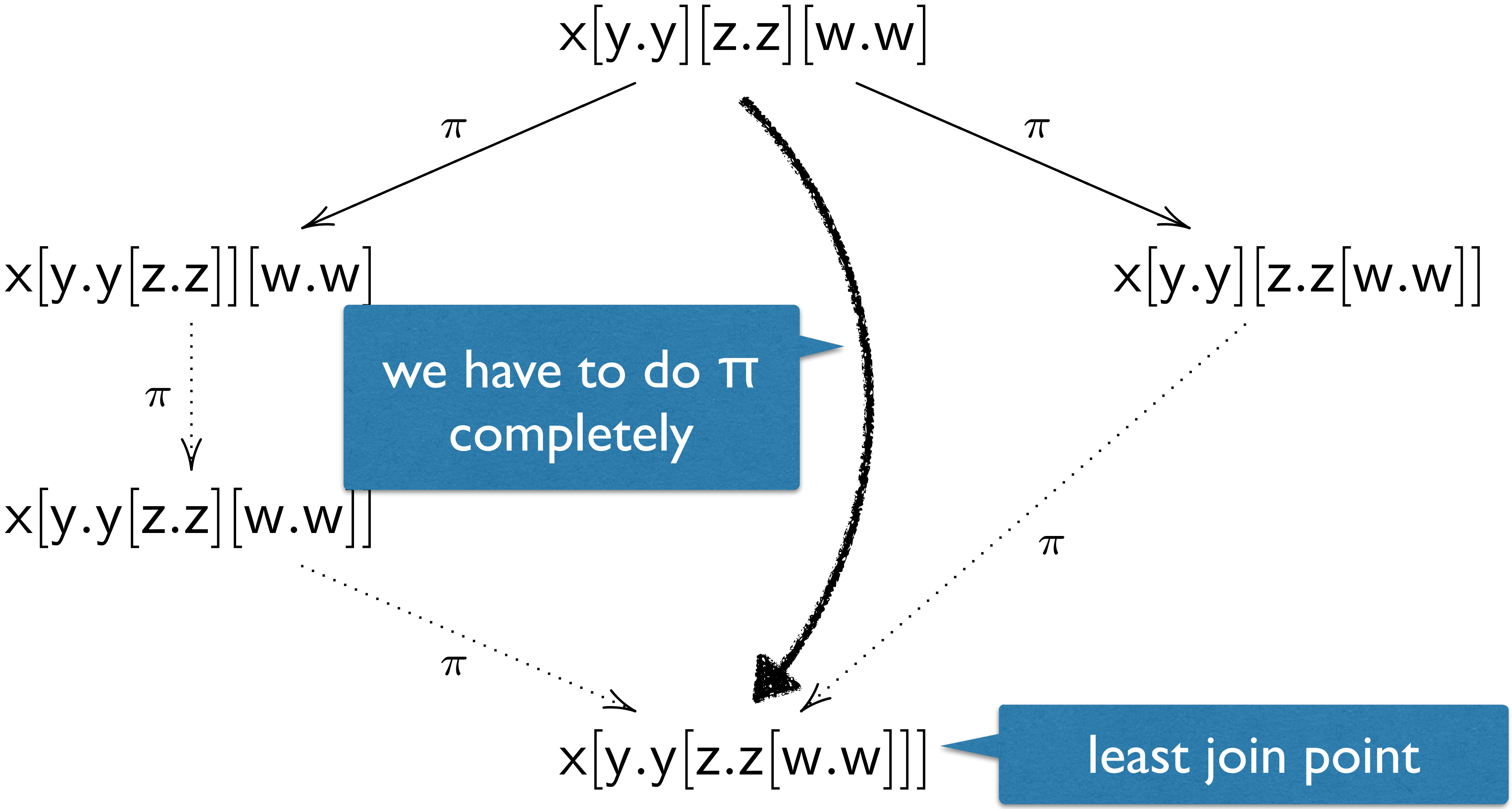
$$x[y.y][z.z][w.w] \Rightarrow x[y.y[z.z[w.w]]]$$

- [Ando 2003] defines the parallel reduction
  by means of the notion of segment trees

- We can avoid it by Z theorem

Z for π?


M
N
M*
N*


x[y.y][z.z][w.w]
π
π
x[y.y[z.z]][w.w]
x[y.y][z.z[w.w]]
we have to do π completely
π
x[y.y[z.z][w.w]]
π
π
x[y.y[z.z[w.w]]]
least join point

Complete permutation

$$(\mathsf{M}[\mathsf{x}.\mathsf{N}])@\mathsf{e} = \mathsf{M}[\mathsf{x}.\mathsf{N}@\mathsf{e}]$$

$$\mathsf{M}@\mathsf{e} = \mathsf{Me} \qquad \text{(otherwise)}$$

Example

$$\mathsf{x}[\mathsf{y}.\mathsf{y}[\mathsf{z}.\mathsf{z}[\mathsf{w}.\mathsf{w}]]]@\mathsf{v} = \mathsf{x}[\mathsf{y}.\mathsf{y}[\mathsf{z}.\mathsf{z}[\mathsf{w}.\mathsf{wv}]]]$$

Z for βπ?

- A naïve definition

$$
\begin{aligned}
x^* &= x \\
(\lambda x.M)^* &= \lambda x.M^* \\
(\iota M)^* &= \iota M^* \\
((\lambda x.M)N)^* &= M^*[x := N^*] \\
((\iota M)[x.N])^* &= N^*[x := M^*] \\
(Me)^* &= M^* @ e^* \qquad \text{(otherwise)}
\end{aligned}
$$

Z for βπ?

- A naïve definition

$$
\begin{aligned}
x^* &= x \\
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\end{aligned}
$$

is not Z

$$
\begin{aligned}
x^* &= x \\
(\lambda x.M)^* &= \lambda x.M^* \\
(\iota M)^* &= \iota M^* \\
((\lambda x.M)N)^* &= M^*[x := N^*] \\
((\iota M)[x.N])^* &= N^*[x := M^*] \\
(Me)^* &= M^* @ e^* \qquad \text{(otherwise)}
\end{aligned}
$$

permutation is done for the result of β

- Monotonicity fails

$$(\iota(x[y.y]))[z.z]w \rightarrow_\pi (\iota(x[y.y]))[z.zw]$$

$$
\begin{aligned}
&((\iota(x[y.y]))[z.z]w)^* \\
&= ((\iota(x[y.y]))[z.z])^* @ w \\
&= (x[y.y]) @ w \\
&= x[y.yw]
\end{aligned}
\qquad
\begin{aligned}
&((\iota(x[y.y]))[z.zw])^* \\
&= (zw)^*[z := x[y.y]] \\
&= x[y.y]w
\end{aligned}
$$

- [Ando 2003] avoids it by the notion of residuals

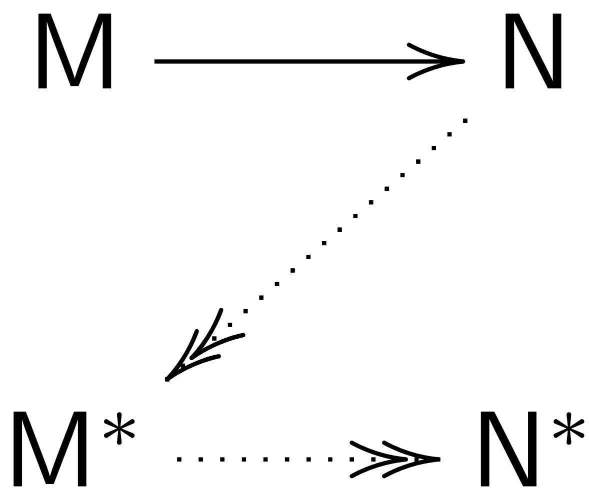
Z for βπ?

- Define a mapping as a composition of
  - M^P = complete permutation
  - M^B = complete development for (only) β
- We want to adapt Z theorem
  to the compositional function M^{PB}

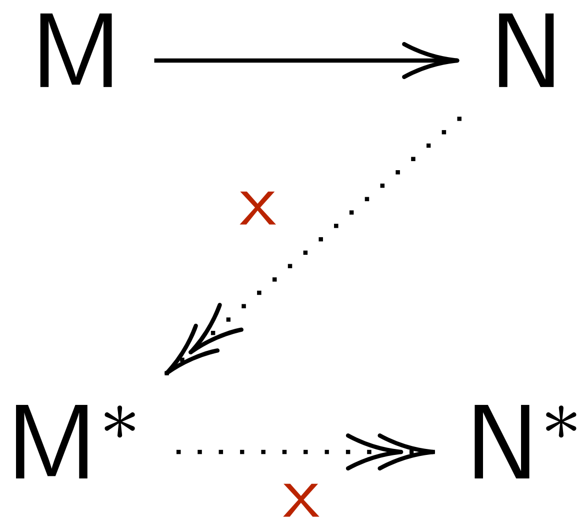
Compositional Z

Z and weak Z

$(\cdot)^*$ is Z for $\to$ iff



$(\cdot)^*$ is weakly Z for $\to$ by $\to_x$ iff



Theorem [N&Fujita2015]

- Let $\to = \to_1 \cup \to_2$

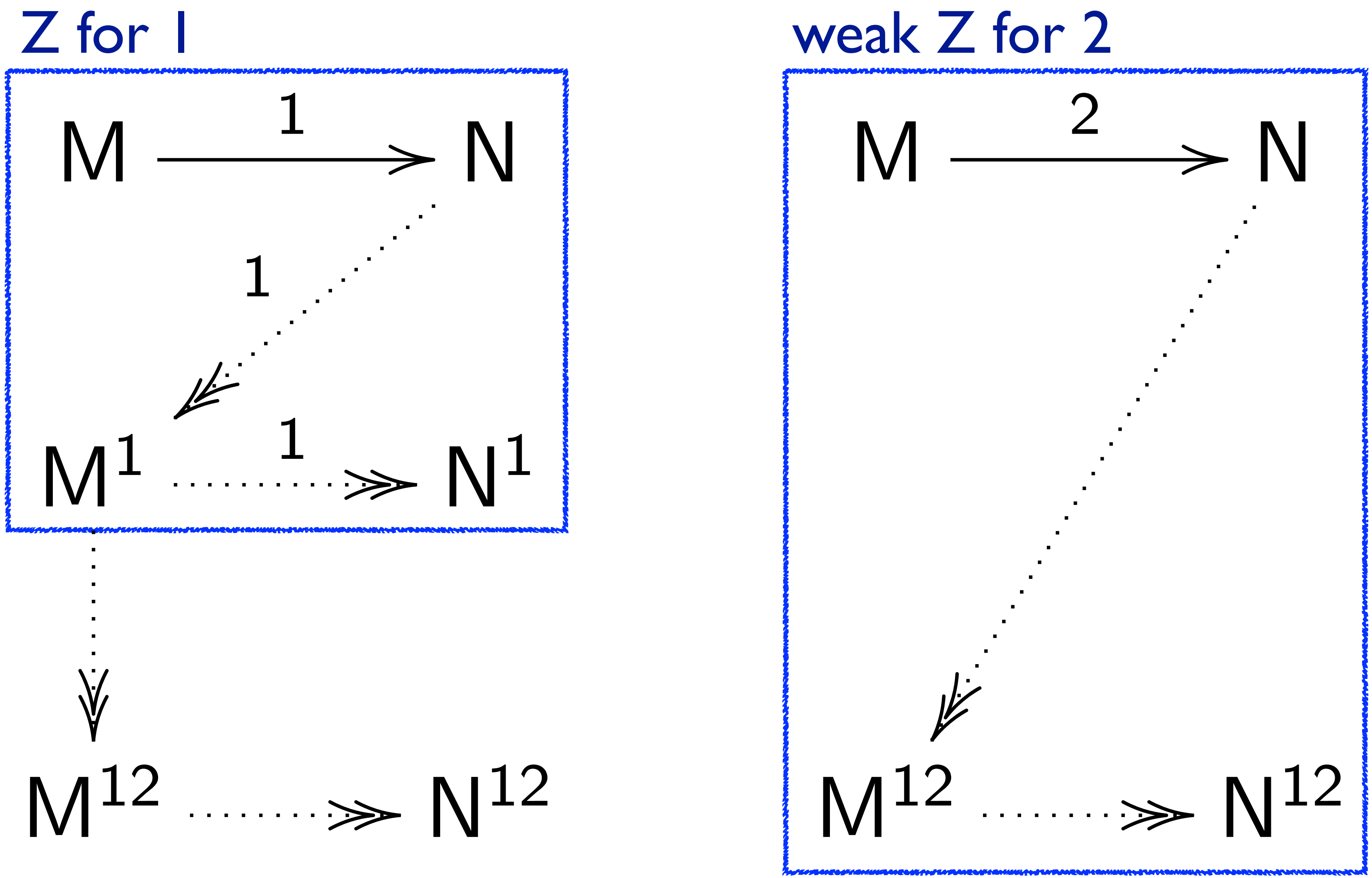
  If mappings $(\cdot)^1$ and $(\cdot)^2$ satisfying following,

  Compositional Z

  - $(\cdot)^1$ is Z for $\to_1$
  - if $M \to_1 N$, then $M^2 \to^* N^2$
  - $M^1 \to^* M^{12}$ holds for any M
  - $(\cdot)^{12}$ is weakly Z for $\to_2$ by $\to$

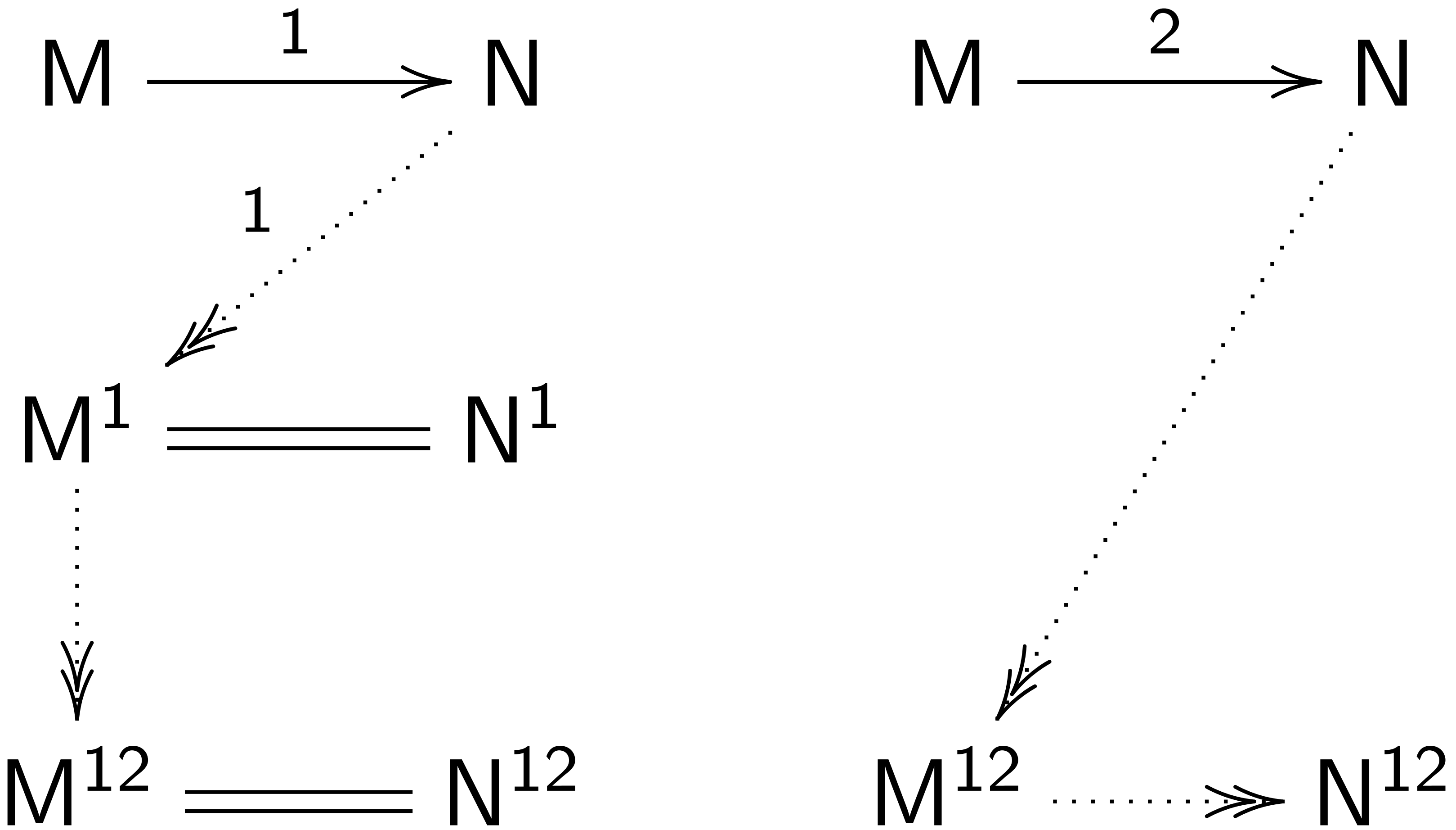
  then the composition $(\cdot)^{12}$ is Z for $\to$

Compositional Z



Compositional Z

If $M \rightarrow_{l} N$ implies $M^{l} = N^{l}$



Corollary [N&Fujita2015]

- If the following hold,
  - if $M \to_1 N$, then $M^1 = N^1$
  - $M \to^* M^1$ holds for any M
  - $M^1 \to^* M^{12}$ holds for any M
  - $(\cdot)^{12}$ is weakly Z for $\to_2$ by $\to$

then the composition $(\cdot)^{12}$ satisfies Z for $\to$

Confluence of βπ by compositional Z

$$
\begin{aligned}
x^P &= x \\
(\lambda x.M)^P &= \lambda x.M^P \\
(\iota M)^P &= \iota M^P \\
(Me)^P &= M^P @ e^P
\end{aligned}
\qquad
\begin{aligned}
x^B &= x \\
(\lambda x.M)^B &= \lambda x.M^B \\
(\iota M)^B &= \iota M^B \\
((\lambda x.M)N)^B &= M^B[x := N^B] \\
((\iota M)[x.N])^B &= N^B[x := M^B] \\
(Me)^B &= M^B e^B \quad \text{(otherwise)}
\end{aligned}
$$

The mappings $(\cdot)^P$ and $(\cdot)^B$ satisfies the conditions of the compositional Z for $\rightarrow_\pi$ and $\rightarrow_\beta$

Extension to λμ

- λμ-calculus [Parigot 1992]
  - corresponds to classical natural deduction
  - λ-calculus extended with control operators
- [Ando 2003] proved confluence of λμ with permutative conversion
  - compositional Z gives simpler proof

$\lambda\mu_{\beta\pi}$

- Terms and eliminators

$$M, N ::= x \mid \lambda x.M \mid \iota M \mid Me \mid \mu\alpha.M \mid [\alpha]M$$

$$e ::= M \mid [x.N]$$

- Reduction rules

$$\begin{array}{rcl} (\lambda x.M)N & \to_\beta & M[x := N] \\ (\iota M)[x.N] & \to_\beta & N[x := M] \\ M[x.N]e & \to_\pi & M[x.Ne] \\ (\mu\alpha.M)e & \to_\mu & \mu\alpha.M[[\alpha]\Box := [\alpha]\Box e] \end{array}$$

λμ and classical logic

double negation elimination

$$\frac{\Gamma \mid \Delta, \alpha : \neg A \vdash M : \bot}{\Gamma \mid \Delta \vdash \mu\alpha.M : A}$$

$$\frac{\Gamma \mid \Delta, \alpha : \neg A \vdash M : A}{\Gamma \mid \Delta, \alpha : \neg A \vdash [\alpha]M : \bot}$$

μ-reduction

$$(\mu\alpha.\mathsf{M})\mathsf{e} \quad \to_\mu \quad \mu\alpha.\mathsf{M}[[\alpha]\Box := [\alpha]\Box\mathsf{e}]$$

$$(\mu\alpha.\cdots[\alpha]\mathsf{P}\cdots)\mathsf{e} \quad \to_\mu \quad \mu\alpha.\cdots[\alpha]\mathsf{Pe}\cdots$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \dfrac{\mathsf{P} : \mathsf{A}\to\mathsf{B}}{[\alpha]\mathsf{P}:\bot}\\ \vdots\\ \mathsf{M}:\bot\end{array}}{\mu\alpha.\mathsf{M} : \mathsf{A}\to\mathsf{B}} \quad \begin{array}{c}\vdots\\ \mathsf{N}:\mathsf{A}\end{array}}{(\mu\alpha.\mathsf{M})\mathsf{N} : \mathsf{B}}$$

just a variant of permutation

$$\xrightarrow{\mu}$$

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathsf{P}:\mathsf{A}\to\mathsf{B}\end{array}\quad\begin{array}{c}\vdots\\ \mathsf{N}:\mathsf{A}\end{array}}{\mathsf{PN}:\mathsf{B}}}{[\alpha]\mathsf{PN}:\bot}\\ \vdots \\ \dfrac{\mathsf{M}[[\alpha]\Box := [\alpha]\Box\mathsf{e}]:\bot}{\mu\alpha.\mathsf{M}[[\alpha]\Box:=[\alpha]\Box\mathsf{e}]:\mathsf{B}}\end{array}$$

λμ and control operators

$$(\mu\alpha.\cdots[\alpha]\mathsf{P}\cdots)\mathsf{e} \quad \to_\mu \quad \mu\alpha.\cdots[\alpha]\mathsf{Pe}\cdots$$

$$(\mu\alpha.\cdots[\alpha]\mathsf{P}\cdots)\underline{\mathsf{N}_1\cdots\mathsf{N}_n} \quad \to_\mu^* \quad \mu\alpha.\cdots[\alpha](\mathsf{P}\underline{\mathsf{N}_1\cdots\mathsf{N}_n})\cdots$$

λμ and control operators

E = [ ]N$_1$…N$_n$
(call-by-name) evaluation context
= continuation

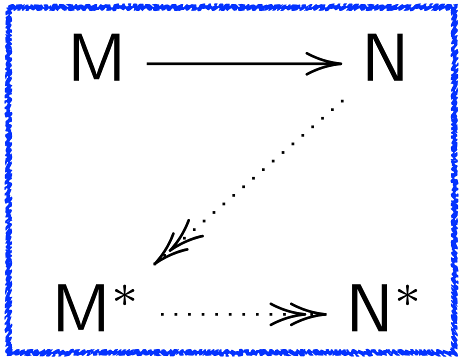
$($ … $\cdots [\alpha]Pe \cdots$

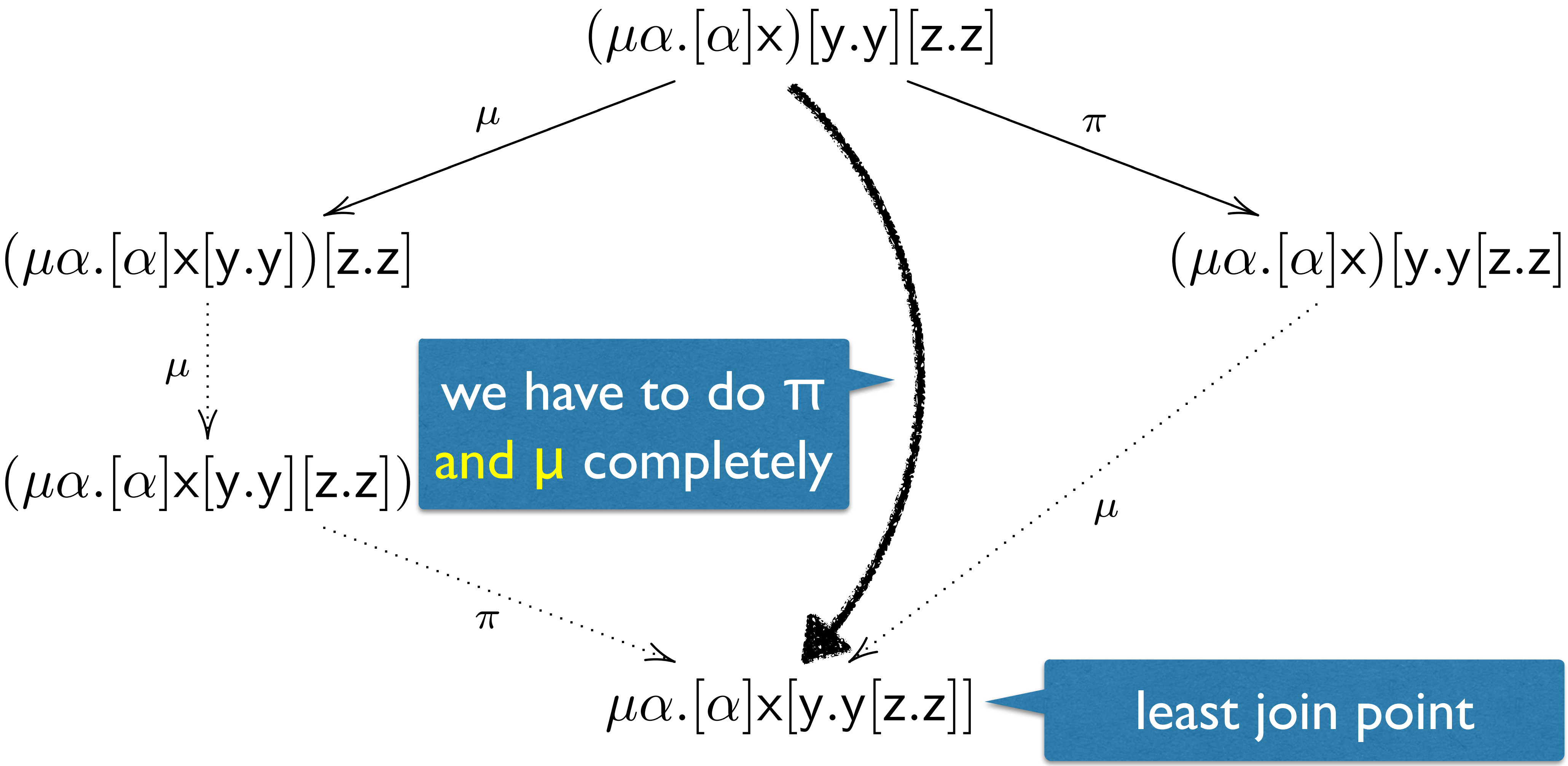
$$(\mu\alpha.\cdots[\alpha]P\cdots)\underline{N_1\cdots N_n} \quad \to^*_\mu \quad \mu\alpha.\cdots[\alpha](P\underline{N_1\cdots N_n})\cdots$$

$$E[\mu\alpha.\cdots[\alpha]P\cdots] \quad \to^*_\mu \quad \mu\alpha.\cdots[\alpha]E[P]\cdots$$

μ captures continuations
(cf. call/cc)

Z for π and μ?


M
N
M*
N*


(μα.[α]x)[y.y][z.z]
μ
π
(μα.[α]x[y.y])[z.z]
(μα.[α]x)[y.y[z.z]]
μ
(μα.[α]x[y.y][z.z])
we have to do π and μ completely
μ
π
μα.[α]x[y.y[z.z]]
least join point

Complete permutation

$$
\begin{aligned}
(\mathsf{M}[\mathsf{x}.\mathsf{N}])@\mathsf{e} &= \mathsf{M}[\mathsf{x}.\mathsf{N}@\mathsf{e}] \\
(\mu\alpha.\mathsf{M})@\mathsf{e} &= \mu\alpha.\mathsf{M}[[\alpha]\square := [\alpha]\square@\mathsf{e}] \\
\mathsf{M}@\mathsf{e} &= \mathsf{Me} \qquad \text{(otherwise)}
\end{aligned}
$$

the definition is
a little complicated

Example

$$
\begin{aligned}
(\mu\alpha.\mathsf{x}[\alpha]\mathsf{y}[\mathsf{z}.\mathsf{z}])@\mathsf{v} &= \mu\alpha.\mathsf{x}[\alpha](\mathsf{y}[\mathsf{z}.\mathsf{z}])@\mathsf{v} \\
&= \mu\alpha.\mathsf{x}[\alpha]\mathsf{y}[\mathsf{z}.\mathsf{zv}]
\end{aligned}
$$

Confluence of βπμ by compositional Z

$$
\begin{aligned}
x^{P} &= x \\
(\lambda x.M)^{P} &= \lambda x.M^{P} \\
(\iota M)^{P} &= \iota M^{P} \\
(\mu\alpha.M)^{P} &= \mu\alpha.M^{P} \\
([\alpha]M)^{P} &= [\alpha]M^{P} \\
(Me)^{P} &= M^{P}@e^{P}
\end{aligned}
$$

$$
\begin{aligned}
x^{B} &= x \\
(\lambda x.M)^{B} &= \lambda x.M^{B} \\
(\iota M)^{B} &= \iota M^{B} \\
(\mu\alpha.M)^{B} &= \mu\alpha.M^{B} \\
([\alpha]M)^{B} &= [\alpha]M^{B} \\
((\lambda x.M)N)^{B} &= M^{B}[x := N^{B}] \\
((\iota M)[x.N])^{B} &= N^{B}[x := M^{B}] \\
(Me)^{B} &= M^{B}e^{B} \quad \text{(otherwise)}
\end{aligned}
$$

The mappings $(\cdot)^{P}$ and $(\cdot)^{B}$ satisfies the conditions of the compositional Z for $\rightarrow_{\pi\mu}$ and $\rightarrow_{\beta}$

Applications of compositional Z

λ with permutative conversion	π and β
λμ with permutative conversion	πμ and β
extensional λ	η and β
λ with explicit subst.	x and β

subst. propagation

Compositional Z enables us to prove confluence by dividing reduction system into two parts

Conclusion

Summary

- Dehornoy and van Oostrom's Z theorem is useful for λ-calculi
- Confluence of λ with permutative conversions becomes much simpler with compositional Z
- Compositional Z suggests (quasi-)modular proofs of confluence

“Simpler” proofs?

- Easier to check?
- Easier to apply other calculi?
- Shorter formal proof? …depending on logical system
- Easier to formalize? …I believe so, but we should check it

“I feel that the new proofs (…) are more beautiful than those we started with, and this is my actual motivation.”

— [Pollack 1995]

A Classical Japanese Poem

composed by Sutoku-In (崇徳院) in 12th cent.

http://www.tamagawa.ac.jp/library/img/h1_077.jpg

瀬を早み 岩にせかるる滝川の
われても末に 逢はむとぞ思ふ

confluence makes us happy!

(direct translation)
A stream of the river separates into two streams
after hitting the rock,
but it will become one stream again

(that is,)
although if I love someone but we cannot be together in this life,
I can be together with her in the next life

Japanese-English Bilingual Corpus of Wikipedia's Kyoto Articles
(National Institute of Information and Communications Technology)