Arrangements of Pseudocircles: On Circularizability*

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Abstract

An arrangement of pseudocircles is a collection of simple closed curves on the sphere or in the plane such that every pair is either disjoint or intersects in exactly two crossing points. We call an arrangement intersecting if every pair of pseudocircles intersects twice. An arrangement is circularizable if there is a combinatorially equivalent arrangement of circles.

Kang and Müller showed that every arrangement of at most 4 pseudocircles is circularizable. Linhart and Ortner found an arrangement of 5 pseudocircles which is not circularizable.

We show that there are exactly four non-circularizable arrangements of 5 pseudocircles, exactly one of them is intersecting. For $n = 6$, we show that there are exactly three non-circularizable digon-free intersecting arrangements. We also have some additional examples of non-circularizable arrangements of 6 pseudocircles.

The claims that we have all non-circularizable arrangements with the given properties are based on a program that generated all connected arrangements of $n \leq 6$ pseudocircles and all intersecting arrangements of $n \leq 7$ pseudocircles. Given the complete lists of arrangements, we used heuristics to find circle representations. Examples where the heuristics failed had to be examined by hand.

1 Introduction

Arrangements of pseudocircles generalize arrangements of circles in the same vein as arrangements of pseudolines generalize arrangements of lines. The study of arrangements of pseudolines was initiated 1918 with an article of Levi [10]. Since then arrangements of pseudolines were intensively studied and the handbook article on the topic [2] lists more than 100 references. The study of arrangements of pseudocircles was initiated by Grünbaum [8].

A *pseudocircle* is a simple closed curve in the plane or on the sphere. An *arrangement of pseudocircles* is a collection of pseudocircles with the property that the intersection of any two of the pseudocircles is either empty or consists of two points where the curves cross. The *graph of an arrangement* $\mathcal{A}$ of pseudocircles has the intersection points of pseudocircles as *vertices*, the vertices split each of the pseudocircles into arcs, these are the *edges* of the graph. Note that this graph may have multiple edges and loop edges without vertices. The graph of an arrangement of pseudocircles comes with a plane embedding, the faces of this embedding are the *cells* of the arrangement. A cell with k crossings on its boundary is a *k-cell*. A 2-cell is also called a *digon* (some authors call it a *lense*), and a 3-cell is also called a *triangle*. An arrangement $\mathcal{A}$ of pseudocircles is

simple, if no three pseudocircles of $\mathcal{A}$ intersect in a common point.
connected, if the graph of the arrangement is connected.

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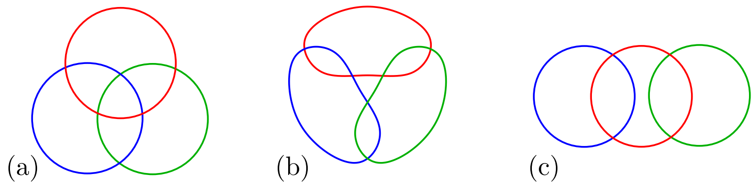

Figure 1 The 3 arrangements of $n = 3$ pseudocircles: (a) *Krupp*, (b) *NonKrupp*, (c) *3-Chain*.

intersecting, if any two pseudocircles of $\mathcal{A}$ intersect.
cylindrical, if there are two cells in $\mathcal{A}$ which are separated by each of the pseudocircles.
Note that every intersecting arrangement is connected. In this paper we assume that arrangements are simple and connected.

Two arrangements $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* if they induce homeomorphic cell decompositions of the plane respectively the sphere. Figure 1 shows the three connected arrangements of three pseudocircles. We call the unique digon-free intersecting arrangement of three (pseudo)circles the *Krupp*[1]. The second intersecting arrangement of three pseudocircles is the *NonKrupp*, this arrangement has digons. The non-intersecting arrangement is the *3-Chain*.

Every triple of great-circles on the sphere induces a Krupp arrangement, hence, we call an intersecting arrangement of pseudocircles an *arrangement of great-pseudocircles* if every subarrangement induced by three pseudocircles is a Krupp.

Some authors think of arrangements of great-pseudocircles when they speak about arrangements of pseudocircles, this is e.g. common practice in the theory of oriented matroids. In fact, arrangements of great-pseudocircles serve to represent rank 3 oriented matroids.

▶ Definition. An arrangement of pseudocircles is *circularizable* if there is an isomorphic arrangement of circles.

Circularizability of arrangements of pseudocircles has not been studied extensively. This paragraph describes the state of the art. Edelsbrunner and Ramos [1] proved non-circularizability of an arrangement of 6 pseudocircles with digons. Linhart and Ortner [11] found a non-intersecting arrangement of 5 pseudocircles with digons which is non-circularizable (Figure 2b). They also proved that every intersecting arrangement of at most 4 pseudocircles is circularizable. Kang and Müller [9] extended the result by showing that all arrangements with at most 4 pseudocircles are circularizable. They also proved that deciding circularizability of connected arrangements is NP-hard. Since stretchability is $\exists\mathbb{R}$-complete, their proof actually implies $\exists\mathbb{R}$-completeness of circularizability.

In our last year's EuroCG contribution [6] we have sketched non-circularizability of two further intersecting arrangements on 5 and 6 pseudocircles, respectively, namely $\mathcal{N}_5^1$ and $\mathcal{N}_6^\Delta$ (see Figures 2a and 3a). Since then, we have extended our results and got the following.

▶ **Theorem 1.1.** *The four equivalence classes of arrangements $\mathcal{N}_5^1$, $\mathcal{N}_5^2$, $\mathcal{N}_5^3$, and $\mathcal{N}_5^4$ (shown in Figure 2) are the only non-circularizable ones among the 984 equivalence classes of connected arrangements of $n = 5$ pseudocircles.*

▶ **Theorem 1.2.** *The three equivalence classes of arrangements $\mathcal{N}_6^\Delta$, $\mathcal{N}_6^2$, and $\mathcal{N}_6^3$ (shown in Figure 3) are the only non-circularizable ones among the 2131 equivalence classes of digon-free intersecting arrangements of $n = 6$ pseudocircles.*

[1] This name refers to the logo of the Krupp AG, a German steel company. Krupp was the largest company in Europe at the beginning of the 20th century. There is also a disease with the German name Pseudo-Krupp, we have no corresponding arrangement.

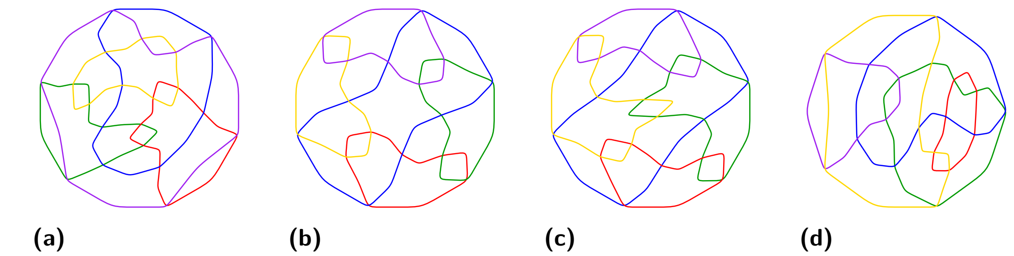

Figure 2 The four non-circularizable arrangements on $n = 5$ pseudocircles. (a) $\mathcal{N}_5^1$. (b) $\mathcal{N}_5^2$. (c) $\mathcal{N}_5^3$. (d) $\mathcal{N}_5^4$.

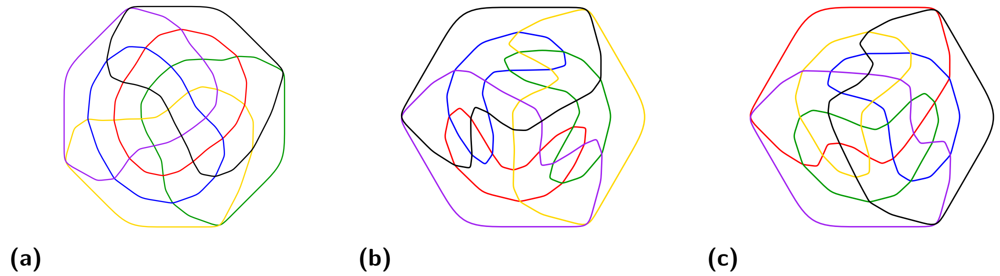

Figure 3 The three non-circularizable digon-free intersecting arrangements for $n = 6$. (a) $\mathcal{N}_6^\Delta$. (b) $\mathcal{N}_6^2$. (c) $\mathcal{N}_6^3$.

Full proofs of Theorems 1.1 and 1.2 can be found in the full version [4], where we also prove non-circularizability of some further interesting arrangements on $n = 6$ pseudocircles and provide some further results for certain classes of arrangements. The non-circularizability proofs use various techniques, most depend on incidence theorems, others use arguments involving metric properties of arrangements of planes, or angles in planar figures.

Our results strongly depend on the generation of the complete lists of connected arrangements of $n \leq 6$ pseudocircles and of intersecting arrangements of $n \leq 7$ pseudocircles. The respective numbers are shown in Table 1. The encoded lists of arrangements up to $n = 6$ are available on our webpage [3]. We remark that the list of intersecting arrangements was already mentioned in our at last year's EuroCG contribution [6]. Computational issues are deferred until Section 5. There we describe the algorithmic ideas behind the computation of the lists.

Particularly interesting is the arrangement $\mathcal{N}_6^\Delta$ (Figure 3a). This is the unique intersecting digon-free arrangement of 6 pseudocircles which attains the minimum 8 for the number of triangles. From our computer search we know that $\mathcal{N}_6^\Delta$ occurs as a subarrangement of every

n	4	5	6	n	4	5	6	7
connected	21	984	609 423	**intersecting**	8	278	145 058	447 905 202
+digon-free	3	30	4 509	+digon-free	2	14	2 131	3 012 972
con.+cylindrical	20	900	530 530	int.+cylindrical		278	144 395	435 367 033
+digon-free		30	4 477	+digon-free			2 131	3 012 906
				great-p.c.s		1	4	11

Table 1 Number of combinatorially different arrangements of n pseudocircles.

digon-free arrangement for $n = 7, 8, 9$ with $p_3 < 2n - 4$ triangles, hence, also neither of those arrangements is circularizable. Therefore, it seems plausible that for every arrangement of n circles $p_3 \geq 2n - 4$. This is the Weak Grünbaum Conjecture. [5, 6]

For the non-circularizability of $\mathcal{N}_6^{\Delta}$ we have two proofs. Due to didactical reasons, we exchanged "first" and "second" in the full version against the actual chronological order.

Our first proof is based on an incidence theorem in 3-space and was already sketched in our last year's EuroCG contribution [6].

Here we sketch our new second proof, which is based on a sweeping argument in 3-D (see Subsection 4). With a similar idea we also show the following theorem, which has some nice corollaries, e.g., it yields a very direct and easy proof that deciding circularizability is $\exists\mathbb{R}$-complete (see Section 3).

▶ **Theorem 1.3** (The Great-Circle Theorem). *An arrangement of great-pseudocircles is circularizable (i.e., has a circle representation) if and only if it has a great-circle representation.*

2 Preliminaries

Stereographic projections map circles to circles (if we consider a line to be a circle containing the point at infinity), therefore, circularizability on the sphere and in the plane is the same concept. Arrangements of circles can be mapped to isomorphic arrangements of circles via Möbius transformations.

Let $\mathcal{C}$ be an arrangement of circles represented on the sphere. Each circle of $\mathcal{C}$ spans a plane in 3-space, hence, we obtain an arrangement $\mathcal{E}(\mathcal{C})$ of planes in $\mathbb{R}^3$. In fact, a fixed sphere S conveys a bijection between (not necessarily connected) circle arrangements on S and arrangements of planes with the property that each plane of the arrangement intersects S.

Consider two circles C_1, C_2 of a circle arrangement $\mathcal{C}$ on S and the corresponding planes E_1, E_2 of $\mathcal{E}(\mathcal{C})$. The intersection of E_1 and E_2 is either empty (i.e., E_1 and E_2 are parallel) or a line ℓ. The line ℓ intersects S if and only if C_1 and C_2 intersect, in fact, $\ell \cap S = C_1 \cap C_2$.

With three pairwise intersecting circles C_1, C_2, C_3 we obtain three planes E_1, E_2, E_3 intersecting in a vertex v of $\mathcal{E}(\mathcal{C})$. It is notable that v is in the interior of the ball bounded by S if and only if the three circles form a Krupp in $\mathcal{C}$.

3 Arrangements of (pseudo) great-circles

Central projections map between arrangements of great-circles on a sphere S and arrangements of lines on a plane. Changes of the plane preserve the isomorphism class of the projective arrangement of lines.

An Euclidean arrangement of n pseudolines can be represented by x-monotone pseudolines, a special representation of this kind is the wiring diagram, see e.g [2]. An x-monotone representation can be glued with a horizontally mirrored copy of itself to form an arrangement of n pseudocircles, see Figure 4. The resulting arrangement is intersecting and has no NonKrupp subarrangement, i.e., it is a great-pseudocircle arrangement.

Indeed the above construction yields a bijection between projective arrangements of n pseudolines in the plane and arrangements of n great-pseudocircles.

Projective arrangements of pseudolines are also known as projective abstract order types or oriented matroids. Their number is known for $n \leq 11$, hence the numbers of great-pseudocircle arrangements given in Table 1 are not new. For more information see [4].

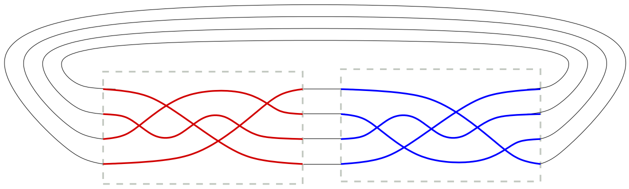

Figure 4 Obtaining an arrangement of pseudocircles from an Euclidean arrangement $\mathcal{A}$ of pseudolines. Arrangement $\mathcal{A}$ and its mirrored copy are shown in red and blue, respectively.

Let $\mathcal{C}$ be an arrangement of great-pseudocircles and let $\mathcal{A}$ be the corresponding projective arrangement of pseudolines. Central projections show that, if $\mathcal{A}$ is realizable with straight lines, then $\mathcal{C}$ is realizable with great-circles, and conversely.

In fact, it is enough that $\mathcal{C}$ is circularizable to conclude that $\mathcal{C}$ is realizable with great-circles and $\mathcal{A}$ is realizable with straight lines.

Proof of Theorem 1.3. Consider an arrangement of circles $\mathcal{C}$ on the unit sphere S that realizes an arrangement of great-pseudocircles. Let $\mathcal{E}(\mathcal{C})$ be the arrangement of planes spanned by the circles of $\mathcal{C}$. Since $\mathcal{C}$ realizes an arrangement of great-pseudocircles, every triple of circles forms a Krupp, hence, the point of intersection of any three planes of $\mathcal{E}(\mathcal{C})$ is in the interior of S.

Imagine the radius of the sphere growing with the time t, to be precise, let $S_1 = S$ and $S_t = t \cdot S$. Since all the intersection points of the arrangement $\mathcal{E}(\mathcal{C})$ are in the interior of S_1, the circle arrangement obtained by intersecting $\mathcal{E}(\mathcal{C})$ with the growing sphere remains the same (isomorphic). Also every circle of the arrangement is moving towards a great-circle while the sphere is growing. When t is large enough it is possible to push all circles a small amount to make them great-circles without changing the arrangement. ◀

▶ **Corollary 3.1.** *Every non-stretchable arrangement of pseudolines has a corresponding non-circularizable arrangement of pseudocircles.*

In particular, the hardness of stretchability directly carries over to hardness of circularizability. Moreover, since there are infinite families of minimal non-stretchable arrangements of pseudolines [7], the same is true for pseudocircles.

It is known that Mnëv's Universality Theorem [12] has strong implications for pseudoline arrangements and stretchability. This together with results from Suvorov [13] directly translates to:

▶ **Corollary 3.2.** *The problem of deciding circularizability is $\exists\mathbb{R}$-complete. Moreover, there exist circularizable arrangements of pseudocircles with a disconnected realization space.*

4 Non-circularizability of $\mathcal{N}_6^\Delta$

Our second proof of non-circularizability of $\mathcal{N}_6^\Delta$ is an immediate consequence of the following theorem, which resembles the proof of the Great-Circle Theorem (Theorem 1.3).

▶ **Theorem 4.1.** *Let $\mathcal{A}$ be a connected digon-free arrangement of pseudocircles with the property that every triple of pseudocircles, which forms a triangles in $\mathcal{A}$, is NonKrupp. Then $\mathcal{A}$ is not circularizable.*

Proof (second proof of non-circularizability of $\mathcal{N}_6^\Delta$). The arrangement $\mathcal{N}_6^\Delta$ is intersecting, digon-free, and each of the eight triangles of $\mathcal{N}_6^\Delta$ is a NonKrupp, hence, Theorem 4.1 implies that $\mathcal{N}_6^\Delta$ is not circularizable. ◀

5 Computational Part

To produce the database of all intersecting arrangements of up to $n = 7$ pseudocircles, we used the dual graphs and a procedure, which generates the duals of all possible extensions by one additional pseudocircle of a given arrangement, starting with the unique arrangement of two intersecting pseudocircles [4, 5]. Another way to obtain the database for a fixed value of n, is to perform a recursive search in the flip graph using the triangle flip operation.

For connected arrangement the dual graph might contain multiple edges. To avoid problems with non-unique embeddings, we modeled connected arrangements with their primal-dual-graphs where vertices, segments, and faces of the arrangement are represented by a vertex in the graph and two vertices share an edge if the corresponding entities are incident and one of them corresponds to an edge. To generate the database of all connected arrangements for $n \leq 6$, we used the fact that the flip graph is connected when triangle flips and digon flips are used. The enumeration was done by a recursive search on the flip graph.

Having generated the database of arrangements of pseudocircles, we were then interested in identifying the circularizable and the non-circularizable ones. To find circle representations we used computer assistance. Examples where our programs failed to find realizations had to be examined by hand. For more information, we refer to the full version [4].

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